



Chapter

2

Counting Methods

► LEARNING GOALS

You will be able to develop your critical thinking skills related to counting by

- Solving problems that involve the Fundamental Counting Principle
- Understanding and simplifying expressions involving factorial notation
- Solving problems that involve permutations
- Solving problems that involve combinations

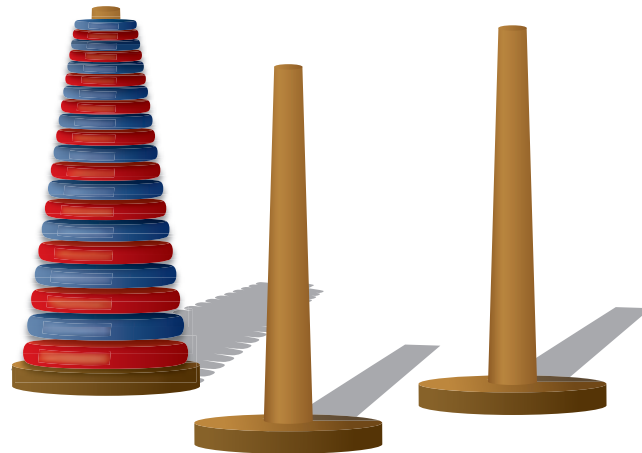
? In how many different orders can these BMX racers finish first, second, and third?

The Tower of Hanoi

The Tower of Hanoi is a puzzle that consists of three vertical rods and a number of disks of different sizes, which can be moved and placed onto any rod. The puzzle starts with all the disks arranged in an orderly stack as shown—the smallest at the top and the largest at the bottom. The goal is to move the entire stack of disks to another rod, following these rules:

- Only one disk may be moved at a time.
- Each move involves taking the upper disk from one of the rods and placing it onto another rod, on top of any other disks that may already be present on that rod.
- No disk may be placed on top of a smaller disk.

The Tower of Hanoi puzzle is called a counting problem. Counting problems involve situations where you determine all the possibilities.



- ?** How can you determine the minimum number of moves needed to complete the puzzle with a stack of 20 disks, without counting every move?
- A.** Suppose you started with a stack consisting of 1 disk. How many moves will it take to move the stack to another rod? What if you started with a stack of 2 disks?
- B.** Try the puzzle, starting with stacks of 1, 2, 3, 4, and then 5 disks. Count and record the minimum number of moves in a table like this:

Number of Disks in Stack	1	2	3	4	5
Minimum Number of Moves					

- C. Compare your completed table with your classmates' tables. Discuss how the disks must be moved so the minimum number of moves are used in each case.
- D. Describe the pattern in your completed table.
- E. Use the pattern to predict the minimum number of moves for stacks of 6 disks to 10 disks, and extend your table to record your predictions.
- F. What type of reasoning did you use to analyze this puzzle: **deductive reasoning** or **inductive reasoning**? How do you know?
- G. Examine your table from part E. How do the numbers in the Minimum Number of Moves row relate to powers of 2? Write an algebraic expression to represent the minimum number of moves needed for a stack of n disks.
- H. Use your expression from part G to determine the minimum number of moves for a stack of 20 disks.

WHAT DO You Think?

Decide whether you agree or disagree with each statement. Explain your decision.

1. The best way to solve a counting problem is to list all the possibilities and then count them. For example, to determine all the different ways a handful of coins can land when flipped all at the same time, list every possible combination of heads and tails and count them.
2. Looking for and extending a pattern is a useful strategy for solving counting problems.
3. There are three people in a group: Sam, Ted, and Jill. Consider these situations:
 - Three people in a line
 - Three people who can wash cars at a fundraiser

When counting the number of ways each situation could occur, Jim claims that the order of the people in the group matters in only one of the situations.



2.1

Counting Principles

YOU WILL NEED

- calculator
- standard deck of playing cards

EXPLORE...

- Suppose you roll a regular red die and a regular blue die at the same time. Describe the **sample space** for this experiment by listing all the different possible outcomes. How many different outcomes are there?

GOAL

Determine the Fundamental Counting Principle and use it to solve problems.

INVESTIGATE the Math

Serge lives in Winnipeg. This summer he plans a sight-seeing trip that includes visiting his family in Regina and Saskatoon. There are many places he might visit on the trip, but he knows he will stop in Regina to visit his brother and then in Saskatoon to visit his parents. He has chosen and mapped out three different routes he can take from Winnipeg to Regina and two different routes he can take from Regina to Saskatoon.



- ?** How does the number of routes he has chosen between Winnipeg and Regina and between Regina and Saskatoon relate to the total number of routes he could take from Winnipeg to Saskatoon?
- Sketch Serge's map and label the highlighted routes from Winnipeg to Regina and Regina to Saskatoon A to E.
 - Complete an **outcome table** to show the sample space for this situation. Count all the possible routes Serge can take from Winnipeg to Saskatoon.

- C. Draw a **tree diagram** to visualize and count all the possible routes he can take from Winnipeg to Saskatoon, through Regina. Compare the results to your answer in part B. What do you notice?
- D. Serge is thinking about adding a fourth route from Winnipeg to Regina, through Dauphin and Yorkton, and a third route from Regina to Saskatoon, through Moose Jaw and Swift Current. Make a **conjecture** about the number of routes he can now take from Winnipeg to Saskatoon via Regina.



- E. Test your conjecture using an organized list, outcome table, or tree diagram.
- F. Make a conjecture about how the number of routes between Winnipeg and Regina and between Regina and Saskatoon relates to the total number of routes he can take from Winnipeg to Saskatoon.
- G. Let a represent the number of routes between Winnipeg and Regina and b represent the number of routes between Regina and Saskatoon. Create an expression that relates a and b to the number of possible routes he can take to go from Winnipeg to Saskatoon.

Reflecting

- H. What type of reasoning did you use to arrive at your conjecture in part D? Explain how you know.
- I. When figuring out the total number of routes Serge can take from Winnipeg to Saskatoon via Regina, which of the following applies?
- You need to consider the number of routes from Winnipeg to Regina AND the number of routes from Regina to Saskatoon.
 - You need to consider the number of routes from Winnipeg to Regina OR the number of routes from Regina to Saskatoon.

What conclusion can you draw?

- J. Discuss the advantages and disadvantages of the three different strategies (outcome table, tree diagram, and calculating) used to solve this counting problem.

APPLY the Math

EXAMPLE 1

Selecting a strategy to solve a counting problem

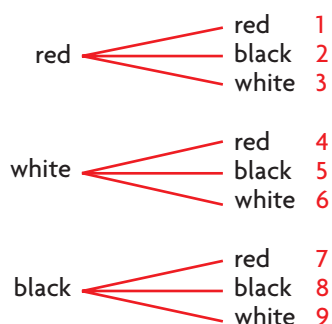
Hannah plays on her school soccer team. The soccer uniform has:

- three different sweaters: red, white, and black, and
- three different shorts: red, white, and black.

How many different variations of the soccer uniform can the coach choose from for each game?

Hannah's Solution: Using a tree diagram

Sweaters Shorts



I created a tree diagram, considering the types of sweaters and shorts that are possible. Each branch of the diagram represents a different variation of the soccer uniform:

1. red sweater and red shorts
2. red sweater and black shorts
3. red sweater and white shorts
4. white sweater and red shorts
5. white sweater and black shorts
6. white sweater and white shorts
7. black sweater and red shorts
8. black sweater and black shorts
9. black sweater and white shorts

There are nine different variations of the soccer uniform to choose from.

I counted the number of branches in my tree diagram.

Mandy's Solution: Using the Fundamental Counting Principle

The number of uniform variations, U , is related to the number of sweaters and the number of shorts:

$$U = (\text{number of sweaters}) \cdot (\text{number of shorts})$$
$$U = 3 \cdot 3 \text{ or } 9$$

Since the coach can choose the type of sweater AND the type of shorts, I knew that I could use the **Fundamental Counting Principle** to calculate the number of uniform variations.

There are nine different variations of the soccer uniform to choose from.

I was able to determine the number of uniform variations without actually counting each one.

Your Turn

- Next season, the coach is adding a fourth style for shorts: red/black stripes. Create an organized list, an outcome table, or a tree diagram to count all the different uniform variations.
- Confirm your count using the Fundamental Counting Principle.

Fundamental Counting Principle

If there are a ways to perform one task and b ways to perform another, then there are $a \cdot b$ ways of performing both.

EXAMPLE 2**Solving a counting problem by extending the Fundamental Counting Principle**

A luggage lock opens with the correct three-digit code. Each wheel rotates through the digits 0 to 9.

- How many different three-digit codes are possible?
- Suppose each digit can be used only once in a code. How many different codes are possible when repetition is not allowed?

**Jake's Solution**

- The number of different codes, C , is related to the number of digits from which to select on each wheel of the lock, D :

$$\begin{aligned} C &= D_1 \cdot D_2 \cdot D_3 \\ C &= 10 \cdot 10 \cdot 10 \\ C &= 1000 \end{aligned}$$

There are 1000 different three-digit codes on this type of lock.

To set the correct code, each wheel must be turned to display the correct digit. I reasoned that I could extend the Fundamental Counting Principle to more than two tasks, since I am selecting a number on wheel A AND selecting a number on wheel B AND selecting a number on wheel C. There are 10 digits that can be selected on each wheel.

- The number of different codes, N , is related to the number of digits from which to select on each wheel of the lock, W :

$$\begin{aligned} N &= W_1 \cdot W_2 \cdot W_3 \\ N &= 10 \cdot 9 \cdot 8 \\ N &= 720 \end{aligned}$$

There are only 720 different three-digit codes when the digits cannot repeat.

I was able to determine the number of codes using the Fundamental Counting Principle.

The problem is almost the same as the one in part a), except each digit can be used only once in each code. That means:

- for the first digit, you have 10 digits to select from, but
- the number of selections for wheel B decreases by 1 to account for the digit used for wheel A, and
- the number of selections for wheel C decreases by 2 to account for the digits used for wheels A and B.

It makes sense that there are fewer codes without repetition, since there are restrictions on the digits that can be selected for wheels B and C.

Your Turn

Suppose you buy a bicycle lock that opens using a five-digit code set by rotating five wheels through the digits 0 to 9. Which lock would be more secure?

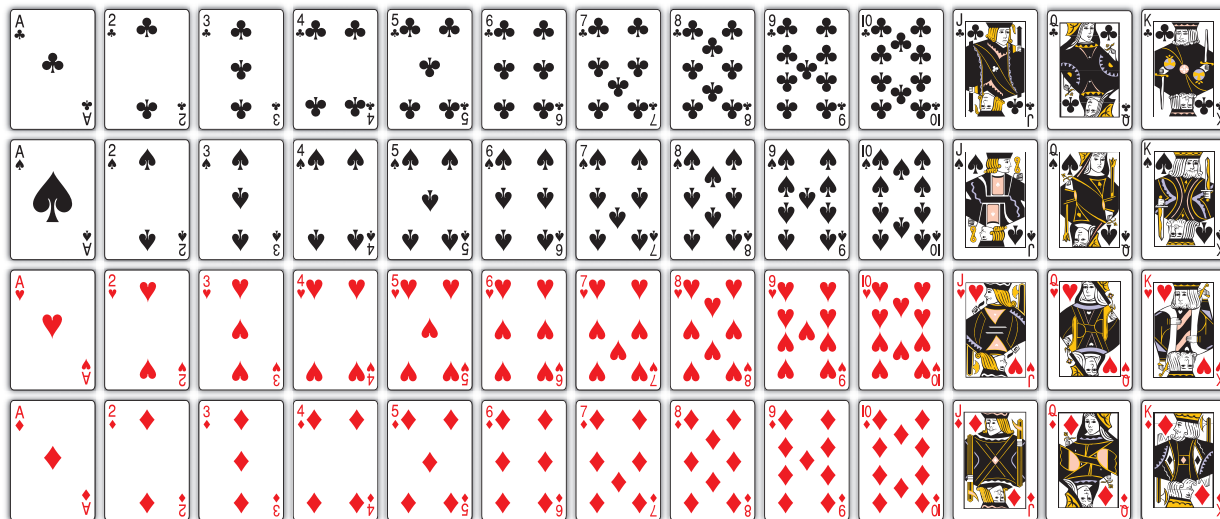
- Lock A, which allows codes with repeating digits
- Lock B, which uses codes that do not allow repeating digits

Explain your answer.

EXAMPLE 3

Solving a counting problem when the Fundamental Counting Principle does not apply

A standard deck of cards contains 52 cards as shown.



Count the number of possibilities of drawing a single card and getting:

- either a black face card or an ace
- either a red card or a 10

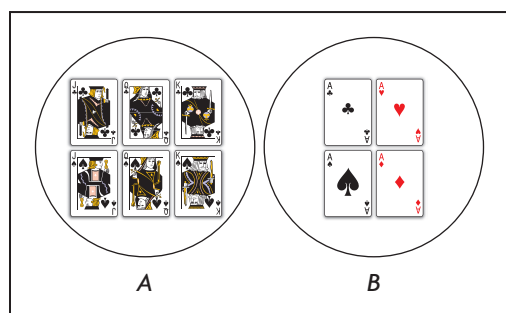
There are four suits, two red and two black, with 13 cards in each suit.

Christian's Solution

- Event A : Draw a black face card.

OR

Event B : Draw an ace.



I let A represent the set of six black face cards and B represent the set of four aces.

There are two events in this situation. I knew that I needed to determine the number of cards in the deck that are either black face cards OR aces.

I drew a Venn diagram to visualize how the two events relate. These events are mutually exclusive.

I knew that I needed to determine the number of elements in the union of two sets with no elements in common.

$$n(A \cup B) = n(A) + n(B)$$

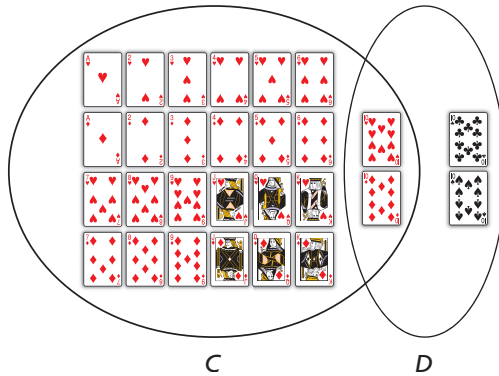
$$n(A \cup B) = 6 + 4$$

$$n(A \cup B) = 10$$

There are 10 ways to draw a single card and get either a black face card or an ace.

I knew I could add the number of ways that each event can occur. I knew I couldn't use multiplication (the Fundamental Counting Principle), since drawing a black face card AND drawing an ace is not possible on a single draw. I counted the number of black face cards and the number of aces in the deck and added.

- b) Event C : Draw a red card.
OR
Event D : Draw a 10.



$$\begin{aligned} n(C \cup D) &= n(C) + n(D) - n(C \cap D) \\ n(C \cup D) &= 26 + 4 - 2 \\ n(C \cup D) &= 28 \end{aligned}$$

I let C represent the set of 26 red cards and I let D represent the set of four 10s.

There are two events in this situation. I knew that I needed to determine the number of cards in the deck that are either red cards OR 10s.

I drew a Venn diagram to visualize how the two events relate. These events are not mutually exclusive. I could see that I had to be careful of overlap, because there are two cards that can be both red and a 10, and I didn't want to count them twice.

I knew that I needed to determine the number of elements in the union of two sets that have common elements.

I knew I could add the number of ways that each event can occur but I had to subtract the number of elements in their intersection, the two red 10s. I added the number of red cards and the number of 10s and then subtracted the number of red 10s.

I used the Principle of Inclusion and Exclusion. I included the elements in the union of both sets and then excluded the elements in their intersection.

There are 28 ways to draw a single card and get either a red card or a 10.

Your Turn

Suppose you want to determine the number of possibilities of drawing a single card and getting:

- Event A : a club or a red card
- Event B : a heart or a 2

How would your strategies be the same? How would they be different?

In Summary

Key Ideas

- The Fundamental Counting Principle applies when tasks are related by the word AND.
- The Fundamental Counting Principle states that if one task can be performed in a ways and another task can be performed in b ways, then both tasks can be performed in $a \cdot b$ ways.

Need to Know

- The Fundamental Counting Principle can be extended to more than two tasks: if one task can be performed in a ways, another task can be performed in b ways, another task in c ways, and so on, then all these tasks can be performed in $a \cdot b \cdot c \dots$ ways.
- The Fundamental Counting Principle does not apply when tasks are related by the word OR. In the case of an OR situation,
 - if the tasks are mutually exclusive, they involve two **disjoint** sets, A and B :

$$n(A \cup B) = n(A) + n(B)$$

- if the tasks are not mutually exclusive, they involve two sets that are not disjoint, C and D :

$$n(C \cup D) = n(C) + n(D) - n(C \cap D)$$

The Principle of Inclusion and Exclusion must be used to avoid counting elements in the intersection of the two sets more than once.

- Outcome tables, organized lists, and tree diagrams can also be used to solve counting problems. They have the added benefit of displaying all the possible outcomes, which can be useful in some problem situations. However, these strategies become difficult to use when there are many tasks involved and/or a large number of possibilities for each task.

CHECK Your Understanding

1. Cam is on vacation. In his suitcase he has three golf shirts (red, blue, and green) and two pairs of shorts (khaki and black).
 - a) Use an outcome table to count the total number of outfit variations he has for golfing.
 - b) Use the Fundamental Counting Principle to verify your result in part a).
2. Missy is buying her first new car. The model she wants comes in four colours (red, black, white, and silver) and she has a choice of leather or cloth upholstery.
 - a) Use a tree diagram to count all the upholstery–colour choices that are available.
 - b) Use the Fundamental Counting Principle to verify your result in part a).
3. For each situation below, indicate whether the Fundamental Counting Principle applies and explain how you know.
 - a) Counting the number of possibilities when rolling a 3 or a 6 with a standard die
 - b) Counting the number of outfit variations when selecting a shirt, a tie, and shoes to wear to the semiformal dance
 - c) Counting the number of possibilities when picking the winner in a stock car race in either the fourth, fifth, or sixth race of the evening
 - d) Counting the number of possibilities to choose from when buying a car with either standard or automatic transmission, air conditioning or not, power windows or not, and GPS navigation or not



PRACTISING

4.
 - a) Kim plays hockey on the Lloydminster Ice Cats. Her team is in a best-two-out-of-three playoff series. Create a tree diagram to show all the win–loss possibilities for her team.
 - b) Use your tree diagram to count the number of ways Kim’s team can win the series despite losing one game.
5. Xtreme clothing company makes snowboarding pants in five colours and sizes of small, medium, large, and extra large. How many different colour–size variations of snowboarding pants does this company make?
6. A computer store sells 5 different desktop computers, 4 different monitors, 6 different printers, and 3 different software packages. How many different computer systems can the employees build for their customers?



7. Jeb's Diner offers a lunch special. You have a choice of 3 soups, 5 sandwiches, 4 drinks, and 2 desserts. How many meals are possible if you choose one item from each category?



8. Tom likes rap music and classic rock. His friend Charlene has 8 rap CDs, 10 classic rock CDs, and 5 country and western CDs in her car. How many CDs can Charlene select from to play in her car stereo that will match Tom's musical tastes?

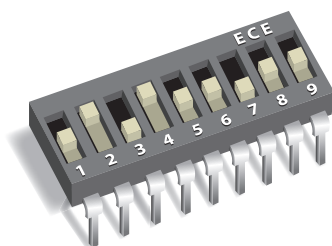


9. Rachelle's bank card has a five-digit PIN where each digit can be 1 to 9.
- How many PINs are possible if each digit can repeat?
 - How many PINs are possible if each digit can be used only once?
10. Computers code information in a binary sequence, using 0 or 1 for each term in the sequence. Each sequence of eight terms is called a byte (for example, 00110010). How many different bytes can be created?
11. a) A country's postal code consists of six characters. The characters in the odd positions are upper-case letters, while the characters in the even positions are digits (0 to 9). How many postal codes are possible in this country?
- b) Canadian postal codes are similar, except the letters D, F, I, O, and U can never appear. (This is because they might be mistaken for the letters E or V or the numbers 0 or 1.) How many postal codes are possible in Canada?

12. A small town in Manitoba has a phone area code of 204 and two different three-digit prefixes as shown: 204-945-□□□□ or 204-940-□□□□. How many different phone numbers are possible for this town?

13. The code to a garage door opener is programmed by moving each of nine switches to any one of three positions. How many different codes are possible?

14. A vehicle rental company has 8 pickup trucks, 10 passenger vans, 35 cars, and 12 sports utility vehicles for rent. How many choices does a customer have when renting just 1 vehicle?



15. The “Pizza Shoppe” offers these choices for each pizza:

- thin or thick crust
- regular or whole-wheat crust
- 2 types of cheese
- 2 types of tomato sauce
- 20 different toppings

How can you determine the number of different pizzas that can be made as follows:

- a) A pizza with any crust, cheese, tomato sauce, and 1 topping
- b) A pizza with a thin whole-wheat crust, tomato sauce, cheese, and no toppings

16. An Alberta licence plate has three letters followed by three digits; for example, ABC 123. The letters I and O are not used to avoid confusion with the digits 1 and 0.

- a) How many different Alberta licence plates are possible?
- b) Because of the growing number of vehicles, the province is changing to plates with three letters followed by four digits; for example, ABC 1234. How many more licence plates are possible?

Closing

17. Counting problems often involve several tasks that are described using the words AND and OR. What is the mathematical meaning behind these words, and how does this affect the strategy you would use to solve counting problems that involve these words? Support your answer using relevant examples.

Extending

18. a) Determine the likelihood that each of the following events can occur using a standard deck of cards.
- i) Drawing a king or a queen
 - ii) Drawing a diamond or a club
 - iii) Drawing an ace or a spade
- b) Does the Fundamental Counting Principle apply to any situation in part a)? Explain.
19. How many two-digit numbers are not divisible by either 2 or 5?
20. A test has 10 true-false questions. A student attempts every question by guessing. What is the likelihood that the student will get a perfect score?
21. Recall Jeb’s Diner and the lunch special from question 7. There are 3 soups, 5 sandwiches, 4 drinks, and 2 desserts to choose from. How many meals are possible if you do not have to choose an item from a category?

2.2

Introducing Permutations and Factorial Notation

YOU WILL NEED

- calculator

EXPLORE...

- How many different single-file lines are possible when Melissa, Nika, and Francois line up at a cashier at a fast-food restaurant?

permutation

An arrangement of distinguishable objects in a definite order. For example, the objects a and b have two permutations, ab and ba .

GOAL

Use factorial notation to solve simple permutation problems.

LEARN ABOUT the Math

Naomi volunteers after school at a daycare centre in Whitehorse, Yukon. Each afternoon, around 4 p.m., she lines up her group of children at the fountain to get a drink of water.



- ? How many different arrangements of children can Naomi create for the lineup for the water fountain if there are six children in her group?

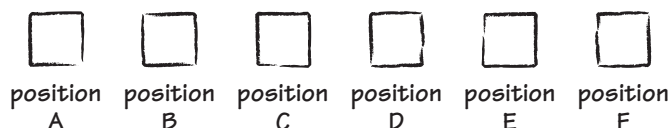
EXAMPLE 1

Solving a counting problem where order matters

Determine the number of arrangements that six children can form while lining up to drink.

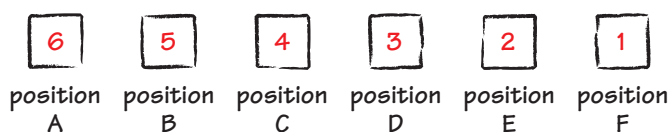
Evgeni's Solution

There are six children in the lineup, so there are six possible positions:



The problem involves arranging six children in a lineup. Each time you change the order of the children in the lineup, you get a different arrangement or **permutation** of the children.

I created a box diagram to help me visualize the problem.



I knew there were six possible children that could be in position A, since there are six children in the group. I then realized there would be only five children available for position B, since one of the six children would already be in position A. The number of possibilities for each position has to decrease by 1 to account for the children placed in previous positions.

Let L represent the total number of permutations:

$$L = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$L = 6!$$

I wrote a product for the total number of permutations possible and then wrote it in a simpler form using **factorial notation**. I can now call the product 6 factorial.

$$L = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$L = 30 \cdot 4 \cdot 6$$

$$L = 120 \cdot 6$$

$$L = 720$$

I used the product form to evaluate 6!. Because of the numbers involved, I was able to use mental math.

There are 720 permutations of the six children at the fountain.

Reflecting

- Why did Evgeni multiply the number of possibilities for each position?
- Why did the factors in Evgeni's product decrease by 1 each time?
- Use n to represent the number of children in a group to be lined up. What expression represents the total number of permutations possible?

factorial notation

A concise representation of the product of consecutive descending natural numbers:
 $n! = n(n - 1)(n - 2) \dots (3)(2)(1)$
 For example:
 $4! = 4 \cdot 3 \cdot 2 \cdot 1$

Communication Tip

Since factorial notation is defined only for natural numbers, expressions like $(-2)!$ and $\frac{1}{2}!$ have no meaning.

APPLY the Math

EXAMPLE 2

Evaluating numerical expressions involving factorial notation

Evaluate the following.

a) $10!$ b) $\frac{12!}{9!3!}$

Connie's Solution: Calculating by hand

a) $10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
 $10! = 3\,628\,800$

I wrote $10!$ as a product. I decided to use a calculator to evaluate the product, since I couldn't figure out a mental math or pencil-and-paper strategy that would be practical.

b) $\frac{12!}{9!3!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)(3 \cdot 2 \cdot 1)}$

I wrote each factorial as a product.

$$\frac{12!}{9!3!} = \frac{12}{3} \cdot \frac{11}{2} \cdot \frac{10}{1} \cdot \frac{9}{9} \cdot \frac{8}{8} \cdot \frac{7}{7} \cdot \frac{6}{6} \cdot \frac{5}{5} \cdot \frac{4}{4} \cdot \frac{3}{3} \cdot \frac{2}{2} \cdot \frac{1}{1}$$

I simplified the expression by aligning common factors in the numerator and denominator. Then I realized I could write part of the expression as $\frac{9!}{9!}$, which is equal to 1.

$$\frac{12!}{9!3!} = \frac{12}{3} \cdot \frac{11}{2} \cdot 10 \cdot \frac{9!}{9!}$$

$$\frac{12!}{9!3!} = \frac{12}{3} \cdot \frac{11}{2} \cdot 10 \cdot 1$$

I simplified further by dividing 12 by 6, the product of the denominators 2×3 .

$$\frac{12!}{9!3!} = 2 \cdot 11 \cdot 10$$

$$\frac{12!}{9!3!} = 22 \cdot 10$$

Since 11 multiplied by 2 is 22, I used mental math to determine the product of 22 and 10.

$$\frac{12!}{9!3!} = 220$$

Your Turn

Create a permutation problem that can be solved by evaluating $10!$.

EXAMPLE 3**Simplifying an algebraic expression involving factorial notation**

Simplify, where $n \in \mathbb{N}$.

a) $(n + 3)(n + 2)!$ b) $\frac{(n + 1)!}{(n - 1)!}$

Phil's Solution

a) $(n + 3)(n + 2)! = (n + 3)[(n + 2)(n + 1)(n)(n - 1)\dots(3)(2)(1)]$

The expression indicates to multiply $(n + 3)$ by the factors of $(n + 2)!$, so I wrote $(n + 2)!$ as a product.

$$\begin{aligned}(n + 3)(n + 2)! &= (n + 3)(n + 2)(n + 1)(n)(n - 1)\dots(3)(2)(1) \\ (n + 3)(n + 2)! &= (n + 3)!\end{aligned}$$

I know that $5! = 5(4!)$, so it makes sense that $n! = n[(n - 1)!]$. I used this reasoning to simplify the product, since $(n + 3)$ and the factors of $(n + 2)!$ are all the factors of $(n + 3)!$.

b) $\frac{(n + 1)!}{(n - 1)!} = \frac{(n + 1)(n)(n - 1)(n - 2)(n - 3)\dots(3)(2)(1)}{(n - 1)(n - 2)(n - 3)\dots(3)(2)(1)}$

I wrote both $(n + 1)!$ and $(n - 1)!$ as products. Then I looked for common factors in the numerator and denominator.

$$\frac{(n + 1)!}{(n - 1)!} = \frac{(n + 1)(n)(n - 1)!}{(n - 1)!}$$

$$\frac{(n + 1)!}{(n - 1)!} = (n + 1)(n)$$

$$\frac{(n + 1)!}{(n - 1)!} = n^2 + n$$

I realized that I could write the expressions in both the numerator and denominator using the common factor $(n - 1)!$.

I simplified the expression using the fact that $\frac{(n - 1)!}{(n - 1)!} = 1$

Your Turn

When $(n + 3)(n + 2)!$ is multiplied by $(n + 4)$ and then divided by $(n + 2)!$, what is the result?

EXAMPLE 4

Solving an equation involving factorial notation

Solve $\frac{n!}{(n-2)!} = 90$, where $n \in \mathbb{I}$.

Katrina's Solution

$$\frac{n(n-1)(n-2)(n-3)\dots(3)(2)(1)}{(n-2)(n-3)\dots(3)(2)(1)} = 90$$

I wrote each factorial as a product.

Then I looked for common factors in the numerator and denominator.

$$\frac{n(n-1)(n-2)!}{(n-2)!} = 90$$

I realized that I could write the expressions in the numerator and denominator using the common factor of $(n-2)!$.

$$n(n-1) = 90$$

$$n^2 - n = 90$$

I simplified the expression using the fact that

$$\frac{(n-2)!}{(n-2)!} = 1$$

$$n^2 - n - 90 = 0$$

$$(n-10)(n+9) = 0$$

$$n-10 = 0 \text{ or } n+9 = 0$$

$$n = 10 \quad n = -9$$

Since the result was a quadratic equation, I rearranged the equation so that the right side was zero. I factored the left side and used each factor to solve the equation.

Check $n = 10$.

Check $n = -9$.

LS	RS	LS	RS
$\frac{10!}{(10-2)!}$	90	$\frac{(-9)!}{(-9-2)!}$	90
$\frac{10!}{8!}$		$\frac{(-9)!}{(-11)!}$	
$\frac{10 \cdot 9 \cdot 8!}{8!}$		is undefined	
$10 \cdot 9$			
90			

I verified my solutions by substituting into the original equation.

Although -9 is an integer, since factorial notation is defined only for natural numbers, $n = -9$ is undefined and therefore is extraneous.

Only $n = 10$ results in the left side equalling the right side, and $10 \in \mathbb{I}$.

There is one solution, $n = 10$.

Your Turn

Solve $\frac{(n+4)!}{(n+2)!} = 6$, where $n \in \mathbb{I}$.

In Summary

Key Ideas

- A permutation is an arrangement of objects in a definite order, where each object appears only once in each arrangement. For example, the set of three objects a , b , and c can be listed in six different ordered arrangements or permutations:

	Position 1	Position 2	Position 3
Permutation 1	a	b	c
Permutation 2	a	c	b
Permutation 3	b	a	c
Permutation 4	b	c	a
Permutation 5	c	a	b
Permutation 6	c	b	a

- The expression $n!$ is called n factorial and represents the number of permutations of a set of n different objects and is calculated as

$$n! = n(n - 1)(n - 2)\dots(3)(2)(1)$$

Need to Know

- In the expression $n!$, the variable n is defined only for values that belong to the set of natural numbers; that is, $n \in \{1, 2, 3, \dots\}$.

CHECK Your Understanding

- Evaluate the following expressions.

a) $6!$ c) $\frac{5!}{3!}$ e) $3! \cdot 2!$
 b) $9 \cdot 8!$ d) $\frac{8!}{7!}$ f) $\frac{9!}{4! \cdot 3!}$

- How many permutations are possible of Ken, Sarah, and Raj when they line up to buy a slice of pizza? Describe your strategy.
 - Express the number of permutations using factorial notation.

- Write the following expressions using factorial notation.

a) $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ c) $\frac{15 \cdot 14 \cdot 13}{4 \cdot 3 \cdot 2 \cdot 1}$
 b) $9 \cdot 8 \cdot 7$ d) $100 \cdot 99$

- Which expressions are undefined? Explain how you know.

a) $(-4)!$ b) $7!$ c) $5.5!$ d) $\frac{3}{4}!$



PRACTISING

5. Evaluate the following expressions.

$$\begin{array}{lll} \text{a)} & 8 \cdot 7 \cdot 6! & \text{c)} \quad \frac{8!}{2! \cdot 6!} & \text{e)} \quad 4\left(\frac{6!}{2! \cdot 2!}\right) \\ \text{b)} & \frac{12!}{10!} & \text{d)} \quad \frac{7 \cdot 6!}{5!} & \text{f)} \quad 4! + 3! + 2! + 1! \end{array}$$

6. Simplify each of the following expressions, where $n \in \mathbb{I}$.

$$\begin{array}{lll} \text{a)} & \frac{n!}{(n-1)!} & \text{c)} \quad \frac{(n+1)!}{n!} & \text{e)} \quad \frac{(n+5)!}{(n+3)!} \\ \text{b)} & (n+4)(n+3)(n+2)! & \text{d)} \quad \frac{n!}{(n-3)!} & \text{f)} \quad \frac{(n-2)!}{(n-1)!} \end{array}$$



The Calgary Stampede, held every July, celebrates Western heritage. The first Stampede was held in 1912.

7. How many different permutations can be created when nine students line up to buy tickets for the afternoon rodeo at the Calgary Stampede?
8. The environmental club has five members. They want to select a president, a vice-president, a secretary, a treasurer, and a spokesperson. How many different ways can this be done?
9. Amy and Traci are planning a summer trip to Vancouver Island in British Columbia. They plan to spend six days in Tofino. While they are there, they have decided to take part in the following activities: whale watching, hiking, surfing, sea kayaking, snorkeling, and fishing. They plan to do a different activity each day. In how many different ways can they sequence these activities over the six days?



10. Visitors to a movie website will be asked to rank 28 movies. The website will present the movies in a different order for each visitor to reduce bias in the poll. How many permutations of the movie list are possible?
11. Solve for n , where $n \in \mathbb{I}$.
- $$\begin{array}{ll} \text{a)} \quad \frac{(n+1)!}{n!} = 10 & \text{c)} \quad \frac{(n-1)!}{(n-2)!} = 8 \\ \text{b)} \quad \frac{(n+2)!}{n!} = 6 & \text{d)} \quad \frac{3(n+1)!}{(n-1)!} = 126 \end{array}$$
12. A baseball coach is determining the batting order for the nine players he is fielding. The coach has already decided that he wants the pitcher to hit in last position. How many different batting orders are possible?

13. On an assembly line for a company that makes digital cameras, seven-digit serial numbers are assigned to each camera under the following conditions:
- Only the digits from 3 to 9 are used.
 - Each digit is to be used only once in each serial number.
- How many different serial numbers are possible? Express your answer using factorial notation and explain why your answer makes sense.
14. A model train has an engine, a caboose, a tank car, a flat car, a boxcar, a livestock car, and a refrigerator car. How many different ways can the cars be arranged between the engine and the caboose?
15. Eight drivers have made it to the final chuckwagon race at Back to Batoche Days, although Brant's wagon is considered certain to win. In how many different orders can the eight chuckwagons finish, if Brant's wagon wins?



Each year for three days in July, Saskatchewan's Métis Nation gathers at the site of the Battle of Batoche for a festival called Back to Batoche Days. The Métis settled this area, northeast of Saskatoon, Saskatchewan, in the early 1870s.

Closing

16. Consider the word YUKON and all the ways you can arrange its letters using each letter only once.
- a) One possible permutation is KYNOU. Write three other possible permutations.
 - b) Use factorial notation to represent the total number of permutations possible. Explain why your expression makes sense.

Extending

17. a) For what values of n is $n!$ greater than 2^n ?
b) For what values of n is $n!$ less than 2^n ?
18. Darlene and Arnold belong to the Asham Stompers, a 10-member dance troupe based in Winnipeg, Manitoba, that performs traditional Métis dances. During the Red River Jig, they always arrange themselves in a line, with Darlene and Arnold next to each other. How many different arrangements of the dancers are possible for the Red River Jig?

2.3

Permutations When All Objects Are Distinguishable

YOU WILL NEED

- calculator
- standard deck of playing cards

GOAL

Determine the number of permutations of n objects taken r at a time, where $0 \leq r \leq n$.

INVESTIGATE the Math

Chloe's school is holding elections for five positions on next year's student council.

EXPLORE...

- How many three-letter permutations can you make with the letters in the word MATH?

OFFICIAL BALLOT

Candidates

Holly Adams	Seung-Hi Park
Bill Mathews	Erica Huzar
Peter Prevc	

<u>PRESIDENT</u>	
<u>VICE-PRESIDENT</u>	
<u>TREASURER</u>	
<u>SECRETARY</u>	
<u>SOCIAL CONVENOR</u>	

- ?** How can Chloe determine the number of ways she can fill out her ballot, if she considers voting for only some of these positions?
- A.** How many different ways can she fill out the ballot if she votes for all five positions? Write your answer using factorial notation.
- B.** Use the Fundamental Counting Principle to write the number of ways she can fill out her ballot for each of the following numbers of positions. Leave your answers in factored form.
- She votes for only the president, vice-president, treasurer, and secretary.
 - She votes for only the president, vice-president, and treasurer.
 - She votes for only the president and vice-president.
 - She votes for only the president.
- C.** Natalie claims that the expression $\frac{5!}{1!}$ represents the number of ways Chloe can fill out her ballot to vote for only the president, vice-president, treasurer, and secretary. Do you agree? Explain.

- D. Use Natalie's idea to write expressions involving the quotient of two factorials for each of the other voting choices in part B.
- E. How do the numbers in each expression in part D relate to the number of positions on the ballot and the number of positions she votes for?
- F. Suppose there are n positions on the student council and Chloe votes for r of these positions, where $r \leq n$. Write an expression that represents the number of ways she could fill out her ballot.

Reflecting

- G. Suppose there are 10 student council positions and she votes for only the president and vice-president. Use the expression you wrote in part F to determine the number of ways Chloe could fill out her ballot. Verify your answer using the Fundamental Counting Principle.
- H. Recall your answer to part A. Does the expression you wrote in part F work for the situation described in part A? Explain.
- I. What voting situation would the expression $\frac{10!}{(10 - 6)!}$ represent?

Communication **Notation**

${}_nP_r$ is the notation commonly used to represent the number of permutations that can be made from a set of n different objects where only r of them are used in each arrangement, and $0 \leq r \leq n$.

When all available objects are used in each arrangement, n and r are equal, so the notation ${}_nP_n$ is used.

APPLY the Math

EXAMPLE 1

Solving a permutation problem where only some of the objects are used in each arrangement

Matt has downloaded 10 new songs from an online music store. He wants to create a playlist using 6 of these songs arranged in any order. How many different 6-song playlists can be created from his new downloaded songs?



Matt's Solution

$${}_nP_r = \frac{n!}{(n-r)!}$$

When $n = 10$ and $r = 6$,

$${}_{10}P_6 = \frac{10!}{(10-6)!}$$

$${}_{10}P_6 = \frac{10!}{4!}$$

$${}_{10}P_6 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{\cancel{4!}}$$

$${}_{10}P_6 = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$$

$${}_{10}P_6 = 151\,200$$

A B C D E F
G H I J A B
D A H F E I
B D J H A E

Choices for each song in the playlist:

10	9	8	7	6	5
song	song	song	song	song	song
1	2	3	4	5	6

Number of different 6-song playlists:

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = {}_{10}P_6$$

There are 151 200 different 6-song playlists that can be created from 10 songs.

I needed to determine the number of different ordered arrangements that can be made from 10 songs, where each arrangement has only 6 songs.

I knew this was the same as determining the number of permutations possible for 10 different objects (n), where each permutation consists of only 6 of those objects (r).

I simplified the expression using the fact that $\frac{4!}{4!} = 1$. Then I multiplied the remaining numbers using a calculator.

I checked to see if my answer was reasonable. I began by representing each of the 10 downloaded songs with a letter from A to J and then wrote some possible arrangements using 6 of these letters to represent each playlist.

I then realized that I could also use the Fundamental Counting Principle. For the first song there are 10 choices, for the second song there are 9 left to choose from, for the third song, only 8 choices are left, and so on. I drew a box diagram to help me visualize this.

I could have used the ${}_nP_r$ formula or the Fundamental Counting Principle to solve this problem.

Your Turn

Determine all the possible 7-song playlists, then 8-song playlists, and finally 9-song playlists that you can create from 10 songs. How does the value of ${}_nP_r$ change as r gets closer to n ? Is this what you would have predicted? Explain.

EXAMPLE 2**Defining 0!**

Use the formula for ${}_nP_r$ to show that $0! = 1$.

Kat's Solution

$${}_nP_r = \frac{n!}{(n-r)!}$$

I wrote the formula for ${}_nP_r$.

$${}_nP_n = \frac{n!}{(n-n)!}$$

I let $n = r$ and performed the substitution so I'd get an expression that contains $0!$ in the denominator.

$${}_nP_n = \frac{n!}{0!}$$

I substituted and then simplified.

$${}_nP_n = n(n-1)(n-2) \dots (3)(2)(1)$$

I knew that the notation ${}_nP_n$ represented the number of permutations that are possible from a group of n different objects using all the objects in each arrangement.

$${}_nP_n = n!$$

I knew by the Fundamental Counting Principle that $n! = n(n-1)(n-2) \dots (3)(2)(1)$, so ${}_nP_n$ must also be equal to $n!$.

Since,

$${}_nP_n = \frac{n!}{0!} \text{ and } {}_nP_n = n!$$

Since both expressions represent ${}_nP_n$, they must be equal to each other by the transitive property.

Then,

$$\frac{n!}{0!} = n!$$

I reasoned that the only way for $\frac{n!}{0!} = n!$ to be true was for $0!$ to be equal to 1.

$$0! = 1$$

Your Turn

State the values of n for which each expression is defined, where $n \in \mathbb{I}$.

a) $(n+3)!$ b) $\frac{n!}{(n+2)!}$

EXAMPLE 3**Solving a permutation problem involving cases**

Tania needs to create a password for a social networking website she registered with. The password can use any digits from 0 to 9 and/or any letters of the alphabet. The password is case sensitive, so she can use both lower- and upper-case letters. A password must be at least 5 characters to a maximum of 7 characters, and each character can be used only once in the password. How many different passwords are possible?

Tania's Solution

Number of characters = $10 + 2(26)$

Number of characters = 62

$${}_nP_r = \frac{n!}{(n-r)!}$$

Case 1: 5-character passwords for
 $n = 62$ and $r = 5$

Case 2: 6-character passwords for
 $n = 62$ and $r = 6$

Case 3: 7-character passwords for
 $n = 62$ and $r = 7$

Total number of passwords = number of
five-character passwords + number of six-character
passwords + number of seven-character passwords

$$\text{Total number of passwords} = {}_{62}P_5 + {}_{62}P_6 + {}_{62}P_7$$

$$\text{Total number of passwords} = 2\,523\,690\,780\,000$$

There are 2 523 690 780 000 different 5-, 6-,
and 7-character passwords using the 62 characters.

I could use any of 10 digits and 52 letters to create a password. Any change in the order of the characters results in a different password, even if the characters are the same.

Since each character can be used only once, I knew that I needed to calculate the number of permutations possible for 62 objects, where each arrangement uses 5 OR 6 OR 7 of those objects. That meant there were three cases to consider. I knew I could use the ${}_nP_r$ formula for each case.

So, I needed to determine the sum of all three cases to determine the total, because a password can consist of 5 OR 6 OR 7 characters.

Since n is large, I used a calculator to enter each case and determine the sum.

Your Turn

Suppose an online magazine requires each subscriber to have a password with exactly 8 characters, using the same requirements for characters described in the problem above. Which is more secure: the social networking website in Example 3 above or the magazine website? Explain.

EXAMPLE 4

Solving a permutation problem with conditions

At a used car lot, seven different car models are to be parked close to the street for easy viewing.

- The three red cars must be parked so that there is a red car at each end and the third red car is exactly in the middle. How many ways can the seven cars be parked?
- The three red cars must be parked side by side. How many ways can the seven cars be parked?

Nick's Solution

- My diagram shows three red cars in positions P1, P4, and P7:



I created a box diagram to visualize the position of each car in the parking lot.

I knew that changing the order of the cars creates a different arrangement.

In this case, the cars in positions P1, P4, and P7 must be red.



I thought about the condition first. Since there are three red cars, I had three ways to place a red car in position P1, two ways for position P4, and one way for position P7.



That left placing the four non-red cars in the remaining positions using the same reasoning: four ways for position P2, three ways for position P3, two ways for position P5, and one way for position P6.

Let A represent the number of arrangements:

$$A = {}_3P_3 \cdot {}_4P_4$$

$$A = 3! \cdot 4!$$

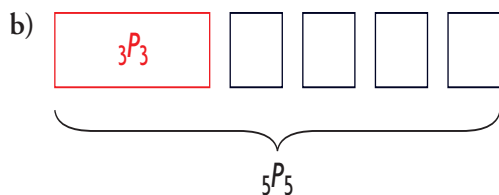
$$A = 6 \cdot 24$$

$$A = 144$$

The cars can be parked 144 different ways.

From the diagram, I could see that I needed to calculate the number of permutations of the three red cars and the number of permutations of the four non-red cars.

I knew I could use the Fundamental Counting Principle to determine the product of the permutations of red and non-red cars, since the red cars must be positioned as shown AND the non-red cars must be positioned as shown to create a line of seven cars.



I used one red box to represent the three red cars because they had to be next to each other, and four boxes for the other four non-red cars.

I knew the three red cars could be arranged $3!$ ways within the red box. Then I thought of the block of red cars as one car in a five-car group.

Let B represent the number of arrangements:

$$B = {}_3P_3 \cdot {}_5P_5$$

$$B = 3! \cdot 5!$$

$$B = 3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$B = 720$$

I knew I could use the Fundamental Counting Principle to multiply the two conditions together because I am arranging the red cars within the red box AND arranging the five boxes (the red box and the four black boxes) to determine the total number of arrangements.

The cars can be parked 720 different ways.

Your Turn

How many ways can the seven cars be parked if the three red cars must be parked side by side and the other four cars must also be side by side?

EXAMPLE 5

Comparing arrangements created with and without repetition

A social insurance number (SIN) in Canada consists of a nine-digit number that uses the digits 0 to 9. If there are no restrictions on the digits selected for each position in the number, how many SINs can be created if each digit can be repeated? How does this compare with the number of SINs that can be created if no repetition is allowed?

Caroline's Solution

Let S represent the number of social insurance numbers if digits are allowed to repeat:

$$S = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$$

I began by determining how many nine-digit arrangements could be made if digits are allowed to repeat.

There are 10 digits from 0 to 9, and I assumed that for each position in the SIN there are 10 possible digits.

$$S = 10^9$$

$$S = 1\,000\,000\,000$$

I noticed that the product formed a power with the number of digits to choose from as the base (n) and the number of those digits actually being used as the exponent (r).

$${}_nP_r = \frac{n!}{(n-r)!}$$

where $n = 10$ and $r = 9$

$${}_{10}P_9 = \frac{10!}{(10-9)!}$$

$${}_{10}P_9 = \frac{10!}{1!}$$

$${}_{10}P_9 = 10!$$

$${}_{10}P_9 = 3\,628\,800$$

$$1\,000\,000\,000 - 3\,628\,800 = 996\,371\,200$$

Therefore, 996 371 200 additional SINS can be made if repetition of digits is allowed.

Then I determined how many nine-digit arrangements could be made if digits are not allowed to repeat.

If each digit can be used only once, then each digit in the arrangement is different. This is the same as determining the number of permutations of 10 objects using 9 in each arrangement.

The difference in the two situations represents the additional SINS that can be made if repetition is allowed.

Your Turn

- In reality, the Canadian government does not use 0, 8, or 9 as the first digit when assigning SINS to citizens and permanent residents, and repetition of digits is allowed. How many nine-digit SINS do not start with 0, 8, or 9?
- SINS starting with the digit 9 are issued to temporary residents. How many SINS are there in total?

In Summary

Key Ideas

- The number of permutations from a set of n different objects, where r of them are used in each arrangement, can be calculated using the formula

$${}_nP_r = \frac{n!}{(n-r)!} \text{ where } 0 \leq r \leq n$$

For example, if you have a set of three objects, a , b , and c , but you use only two of them at a time in each permutation, the number of permutations is

$${}_3P_2 = \frac{3!}{(3-2)!} \text{ or } 6$$

	Position 1	Position 2
Permutation 1	a	b
Permutation 2	a	c
Permutation 3	b	a
Permutation 4	b	c
Permutation 5	c	a
Permutation 6	c	b

- When all n objects are used in each arrangement, n is equal to r and the number of arrangements is represented by ${}_nP_n = n!$.
- The number of permutations that can be created from a set of n objects, using r objects in each arrangement, where repetition is allowed and $r \leq n$, is n^r . For example, the number of four-character passwords using only the 26 lower-case letters, where letters can repeat, is $26 \cdot 26 \cdot 26 \cdot 26 = 26^4$.

Need to Know

- If order matters in a counting problem, then the problem involves permutations. To determine all possible permutations, use the formula for ${}_nP_n$ or ${}_nP_r$, depending on whether all or some of the objects are used in each arrangement. Both of these formulas are based on the Fundamental Counting Principle.

- By definition,

$$0! = 1$$

As a result, any algebraic expression that involves factorials is defined as long as the expression is greater than or equal to zero. For example, $(n+4)!$ is only defined for $n \geq -4$ and $n \in \mathbb{I}$.

- If a counting problem has one or more conditions that must be met,
 - consider each case that each condition creates first, as you develop your solution, and
 - add the number of ways each case can occur to determine the total number of outcomes.

CHECK Your Understanding

- Evaluate the following expressions.
a) ${}_5P_2$ c) ${}_{10}P_5$ e) ${}_7P_7$
b) ${}_8P_6$ d) ${}_9P_0$ f) ${}_{15}P_5$
- Katrina, Jess, Nazir, and Mohamad are running for student council.
a) Create a list or use a tree diagram to count all the ways that a president and vice-president can be elected.
b) Verify your results from part a) using the formula for ${}_nP_r$.
- How many different ways can six different chocolate bars be distributed to each set of children, if each child is to receive only one?
a) 4 children b) 1 child
- Without calculating, predict which value is larger: ${}_{10}P_8$ or ${}_{10}P_2$. Explain your prediction.

PRACTISING

- Nine girls are available to fill three positions on a hockey team: centre, winger, and defence. If all the girls are equally competent in each position, how many different ways can the positions be filled?
- The photo club must select an executive committee to fill the roles of president, vice-president, treasurer, and secretary. All 15 members of the club are eligible. How many different executive committees are possible?
- The MV UMIYAVUT (umiavut means “our boat” in Inuktitut) is a cargo ship that sails the Northwest Passage. If it carries eight different coloured signal flags, how many different signals could be created by hoisting all eight flags on the ship’s flagpole in different orders?
- A wheelchair basketball team sold 5000 tickets in a draw to raise money. How many ways can the tickets be drawn to award the first, second, and third prizes if each winning ticket drawn is not put back prior to the next draw?
- If a Canadian social insurance number begins with the digit 6, it indicates that the number was registered in Manitoba, Saskatchewan, Alberta, Northwest Territories, or Nunavut. If the number begins with a 7, the number was registered in British Columbia or the Yukon. How many different SINS can be registered in each of these groups of provinces and territories?





10. Sandy belongs to the varsity basketball team. There are 12 students on the team. How many ways can the coach select each of the following?
 - a) The starting five players (a point guard, a shooting guard, a small forward, a power forward, and a centre)
 - b) The starting five players, if the tallest student must start at the centre position
 - c) The starting five players, if Sandy and Natasha must play the two guard positions
11. For each expression, state the values of n for which the expression is defined.

a) $\frac{n!}{(n-1)!}$ b) $(n+4)(n+3)(n+2)!$	c) $\frac{(n+1)!}{n!}$ d) $\frac{(n+5)!}{(n+3)!}$
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12. There are six different marbles in a bag. Suppose you reach in and draw one at a time, and do this four times. How many ways can you draw the four marbles under each of the following conditions?
 - a) You do not replace the marble each time.
 - b) You replace the marble each time.
 - c) Compare your answers for parts a) and b). Does it make sense that they differ? Explain.
13. How many ways can five different graduation scholarships be awarded to 20 students under each of these conditions?
 - a) No student may receive more than one scholarship.
 - b) There is no limit to the number of scholarships awarded to each student.
14. All phone numbers consist of a three-digit area code, a three-digit exchange, and then a four-digit number. A town uses the 587 area code and the exchange 355; for example, 587-355-1234. How many different phone numbers are possible in this town under the following conditions?
 - a) There are four different digits in the last four digits of the phone number.
 - b) At least one digit repeats in the last four digits of the phone number.
15. Solve each equation for n . State any restrictions on n .

a) ${}_nP_2 = 20$	b) ${}_{n+1}P_2 = 72$
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16. Solve each equation for r . State any restrictions on r .

a) ${}_6P_r = 30$	b) $2({}_7P_r) = 420$
-------------------	-----------------------
17. Show that for all values of n , where $n \in \mathbb{N}$,

$${}_nP_n = {}_nP_{n-1}$$

Closing

18. a) Describe how the formulas for calculating permutations, ${}_nP_n$ and ${}_nP_r$ where $r \leq n$, are the same and how they are different.
- b) Provide an example of a counting problem that could be solved using each formula.

Extending

19. Five cards are drawn from a standard deck of cards and arranged in a row from left to right in the order in which they are drawn. Determine each of the following.

- a) The number of possible arrangements
- b) The likelihood that an arrangement contains black cards only
- c) The likelihood that an arrangement contains hearts only



20. Show that

$${}_{n+1}P_2 - {}_nP_1 = n^2$$

21. Show that for natural numbers n and whole numbers r , where $n > r$,

$${}_nP_{r+1} = (n-r){}_nP_r$$

Math in Action

Birthday Permutations

The days of the year (not including leap years) can be labelled 1 to 365. You can create a four-number sequence in ascending order to represent the birthdays of four people. For example, 3, 45, 230, 310 represents the birthdays of four people born on the 3rd, 45th, 230th, and 310th days of the year.

- a) Create groups of four. Record the birthdates of everyone in your group as an ascending sequence of numbers from 1 to 365.
- b) Use what you know about permutations to calculate the number of different birthday sequences that are possible for any four-person group.
- c) Use your answer from part b) and what you know about permutations to calculate
- the percent of four-person birthday sequences that have four different birthdays
 - the percent of four-person birthday sequences that have two or more birthdays on the same day
- d) Use actual data collected from the groups in your classroom to calculate
- the percent of four-person birthday sequences with four different birthdays
 - the percent of four-person birthday sequences with two or more birthdays on the same day
- e) How do your calculations for part c) compare to your findings for part d)?



2

Mid-Chapter Review

FREQUENTLY ASKED Questions

Study Aid

- See Lesson 2.1, Examples 1 to 3.
- Try Mid-Chapter Review Questions 1 to 4.

Q: What is the Fundamental Counting Principle and what is it used for?

A: The Fundamental Counting Principle is used to calculate the number of ways a set of tasks can occur in a counting problem. The product of the number of ways each task can occur results in the number of ways that all the tasks can occur. For example:

A breakfast menu offers four choices of eggs (sunny side up, scrambled, poached, soft boiled), two choices of toast (white, whole wheat), and three choices of meat (bacon, sausage, ham). If you must order one item from each category, the number of possible breakfasts, B , is

$$B = 4 \cdot 2 \cdot 3, \text{ or } 24$$

Look for the connecting word AND between each task to identify when the Fundamental Counting Principle should be used.

If a problem involves two or more tasks connected by the word OR, the Fundamental Counting Principle does not apply. In this type of problem, the number of ways each task can occur are added together.

Q: What is factorial notation and what does $n!$ represent?

A: Factorial notation is a short form for representing the product of decreasing consecutive natural numbers.

$$n! = (n)(n-1)(n-2)\dots(3)(2)(1)$$

The expression $n!$ can be used to calculate the number of ways you can make ordered arrangements, or permutations, using n different objects. For example, the number of arrangements of five people in a line, A , is

$$A = 5!$$

$$A = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1, \text{ or } 120$$

Q: How do you determine the number of permutations possible from a group of different objects when you do not use all of them in each permutation?

A: Suppose you have a group of n different objects, and you need to create ordered arrangements using only r of those objects. You can calculate the number of ways this can be done using the formula

$${}_nP_r = \frac{n!}{(n-r)!}$$

For example, the number of ways that a president and vice-president can be selected from a group of 10 candidates is

$${}_{10}P_2 = \frac{10!}{(10-2)!}$$

Study Aid

- See Lesson 2.2, Examples 1 to 4.
- Try Mid-Chapter Review Questions 5 to 8.

Study Aid

- See Lesson 2.3, Examples 1 to 5.
- Try Mid-Chapter Review Questions 9 to 12.

PRACTISING

Lesson 2.1

1. A sub shop offers the following choices:

- 3 types of buns
- 5 types of cold cuts
- 3 types of cheese
- 12 different toppings
- 3 different sauces

If Mario wants a sub with one item from each category, how many different subs can he choose from?

2. Radio and television stations in the USA and Canada have four-character station names. There are 3 choices for the first letter: K, W, and C, and a blank may be used as the last character. For instance, both CKLW and WHO are acceptable names. How many station names are possible?
3. How many ways can a sum of 2 or a sum of 10 be rolled with a pair of dice?



4. How many ways can you select 3 horses to come first, second, and third in a 10-horse race?



Lesson 2.2

5. Evaluate each expression.

a) $8!$ b) $6! \cdot 3!$ c) $\frac{9!}{6!}$ d) $\frac{10 \cdot 9!}{5 \cdot 8!}$

6. How many different lineups can be formed by nine players on a softball team?

7. Simplify each expression.

a) $(n + 5)(n + 4)!$ c) $\frac{(n - 4)!}{(n - 5)!}$
 b) $\frac{(n + 4)!}{(n + 2)!}$ d) $\frac{(n + 2)!}{n!}$

8. Solve each equation.

a) $\frac{n!}{(n - 2)!} = 72$ b) $\frac{(n - 1)!}{(n - 3)!} = 30$

Lesson 2.3

9. Evaluate the following expressions.

a) ${}_9P_2$ b) ${}_{12}P_8$ c) ${}_5P_5$ d) ${}_{12}P_{10}$

10. State the restrictions on the variable, n , for each expression in

- a) question 7
 b) question 8

11. Rennie has 20 CDs in his car. His CD player holds 6 CDs. How many different ways can he load his CD player?

12. Manny is the captain of the 15-member soccer team that has won his city's championship. How many ways can Manny and 2 other players line up to receive the championship trophy, if the captain must be first in line?



13. Margo claims that for some counting problems, you can use either the Fundamental Counting Principle or the permutations formula,

$${}_nP_r = \frac{n!}{(n - r)!}$$

Do you agree? Use examples to support your position.

2.4

Permutations When Objects Are Identical

YOU WILL NEED

- calculator

EXPLORE...

- Consider the words BIKE and BOOK. How does the number of ways that the four letters in each word can be arranged differ? Why?

GOAL

Determine the number of permutations when some objects are identical.

INVESTIGATE the Math

Tom is putting away the dishes after supper. In addition to putting away the bowls, cups, and cutlery, he has to stack seven dinner plates. Three of the plates are white and identical, while the remaining four plates are red, green, yellow, and blue.



- ?** How many different ways can he stack the seven plates to store them in the cupboard?
- If all seven plates are different, in how many ways could he stack the plates?
 - Tom decided to think about the plates as if they are different. To do this, he represented the three identical white plates using three different letter codes: W_1 , W_2 , and W_3 . He then used R for the red plate, G for the green plate, Y for the yellow plate, and B for the blue plate. List all the ways the plates could be stacked if the three white plates are stacked on top of the four coloured plates, and the four coloured plates are stacked in the order red, green, yellow, blue.
 - Examine your list in part B. Recognizing that the white plates are, in fact, identical, how many arrangements in your list are really the same arrangement? How does this number relate to the number of white plates?
 - Use your answers for parts A and C to write an expression that represents the number of different ways these seven plates can be stacked. Use your expression to calculate the number.



Reflecting

- E. Suppose Tom had to stack these plates:



Write an expression to represent the number of different ways to stack the plates.

- F. Suppose Tom had to stack these plates:



Write an expression to represent the number of different ways to stack the plates.

- G. The next time Tom puts away the dishes, the number of arrangements that are possible when he stacks the plates is represented by the expression

$$\frac{10!}{2! \cdot 3! \cdot 4!}$$

How many plates will he be stacking and what colours might they be?

- H. Write an expression to represent the number of permutations of n objects, where a of the objects are identical, another b are identical, and another c are identical.

APPLY the Math

EXAMPLE 1 Solving a permutation problem where objects are alike

In the mountainous regions of India, China, Nepal, and Bhutan, it is common to see prayer flags. Each flag has a prayer written on it, and colour is used to symbolize different elements: green (water), yellow (earth), white (air/wind), blue (sky/space), and red (fire).

How many different arrangements of the same prayer can Dorji make using these 9 flags: 1 green, 1 yellow, 2 white, 3 blue, and 2 red?



Dorji's Solution

Let A represent the number of arrangements of 9 flags:

$$A = \frac{9!}{2! \cdot 3! \cdot 2!}$$

$$A = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3!}{2 \cdot 1 \cdot 3! \cdot 2 \cdot 1}$$

$$A = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$$

$$A = 15\,120$$

There are 15 120 different prayer flag arrangements.

If the 9 flags were different, I could arrange them $9!$ ways. However, some are identical. Since there are 2 white flags, 3 blue flags, and 2 red flags, I divided $9!$ by $2! \cdot 3! \cdot 2!$ to eliminate the arrangements that are the same and would be counted more than once otherwise. For example, if I didn't divide by $2!$ for the 2 identical white flags, these arrangements would be counted as two different arrangements instead of one:

G Y B₁ B₂ B₃ W₁ W₂ R₁ R₂
 G Y B₁ B₂ B₃ W₂ W₁ R₁ R₂

I simplified using the facts that

$$\frac{3!}{3!} = 1 \text{ and } \frac{4}{2 \cdot 1 \cdot 2 \cdot 1} = 1$$

Your Turn

Suppose there are 9 flags, but 3 are white, 3 are red, and 3 are green. Predict whether there would be more or fewer than 15 120 arrangements of these flags. State any assumptions you are making. Verify your prediction by determining the number of arrangements.

EXAMPLE 2**Solving a conditional permutation problem involving identical objects**

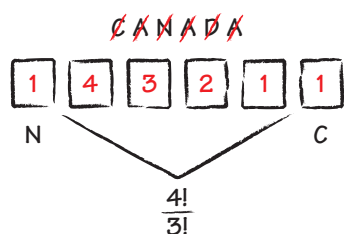
How many ways can the letters of the word CANADA be arranged, if the first letter must be N and the last letter must be C?

Sung Ho's Solution

I drew a box diagram to help me organize my solution.

I knew that each arrangement would contain six letters, and there are two conditions to be met: the first letter in each arrangement had to be N and the last letter C.

Since there is only one of each of these letters, these positions could be filled only one way.



That left four letters left to place. I knew these could be arranged in $4!$ ways if they are all different, but they aren't. The three A's are identical.

I divided $4!$ by $3!$ to eliminate the arrangements that would be the same.

Let A represent the number of arrangements:

$$A = 1 \cdot \frac{4!}{3!} \cdot 1$$

$$A = 1 \cdot 4 \cdot 1$$

$$A = 4$$

I calculated the total number of arrangements using the Fundamental Counting Principle because I am placing the N in the first position AND placing the C in the last position AND arranging the remaining four letters in the four positions in between.

There are four ways the letters in CANADA can be arranged with N as the first and C as the last letter.

NDAAAC

NADAAC

NAADAC

NAAADC

I created an organized list to check my answer.

Since the N and C must be in the first and last positions, I knew the three A's and the D must be in between. The only way I could create different arrangements was to change the position of the D.

My answer seems reasonable, since I was able to list only four different possibilities.

Your Turn

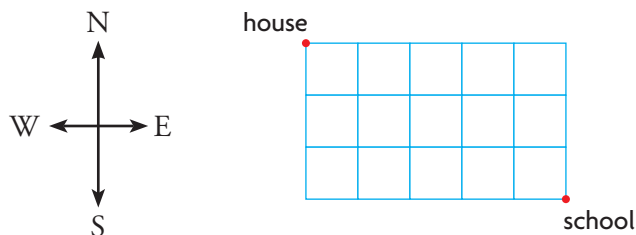
How many different arrangements are there for the six letters in the word CANADA for each situation?

- If there are no conditions for where letters must be placed
- If the first letter has to be C

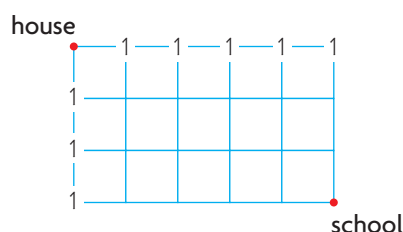
EXAMPLE 3

Solving a permutation problem involving routes

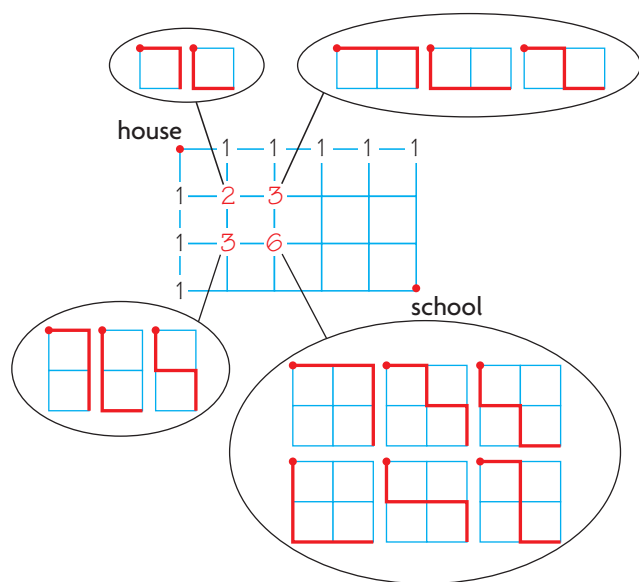
Julie's home is three blocks north and five blocks west of her school. How many routes can Julie take from home to school if she always travels either south or east?



Julie's Solution: Using a diagram



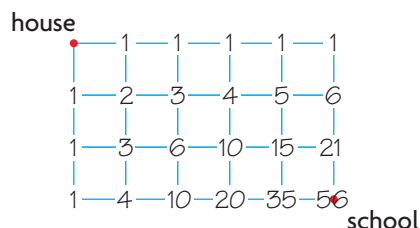
I drew a diagram that indicated the number of ways I could get to each intersection along the perimeter by going just south or just east.



Then I started to count the number of ways I could get to other intersections by going both south and east.

I noticed a pattern; the sum of the numbers on the top right and bottom left corners of each block is equal to the number of routes to the bottom right corner of each block.

$$1 + 1 = 2, 1 + 2 = 3, 2 + 1 = 3, \text{ and } 3 + 3 = 6$$



Using the pattern, I completed the diagram.

There are 56 different routes from home to school travelling south or east.



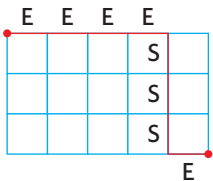
Jean's Solution: Using permutations

Possible routes:



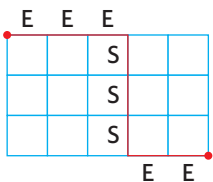
EEEEESSS

Since her school is five blocks east and three blocks south of her home, any route she takes will involve travelling east five blocks and south three blocks.



EEEESSSE

I started to create a list of possible routes using the letter E for east and S for south.



EEESSSEE

I stopped when I realized that this is just a permutation problem involving identical objects. I decided to figure out how many different arrangements there are for the five E's and three S's.

Let R represent the number of routes:

$$R = \frac{8!}{5! \cdot 3!}$$

To determine the number of routes, I needed to determine the number of permutations of eight letters, where five are E's and three are S's.

$$R = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{5! \cdot 3!}$$

$$R = 8 \cdot 7$$

$$R = 56$$

I simplified using the facts that

$$\frac{5!}{5!} = 1 \text{ and } \frac{6}{3!} = 1$$

There are 56 different routes from home to school travelling south or east.

Your Turn

The school is three blocks east and four blocks south of Carrie's house. Predict whether Carrie will have more or fewer than 56 possible routes if she always travels south or east. Determine the number of routes to verify.

In Summary

Key Ideas

- There are fewer permutations when some of the objects in a set are identical compared to when all the objects in a set are different. This is because some of the arrangements are identical.
- The number of permutations of n objects, where a are identical, another b are identical, another c are identical, and so on, is

$$P = \frac{n!}{a!b!c!\dots}$$

For example, in the set of four objects a, a, b , and b , the number of different permutations, P , is

$$P = \frac{4!}{2! \cdot 2!}$$

$$P = 6$$

The six different arrangements are $aabb, bbaa, abab, baba, abba$, and $baab$.

Need to Know

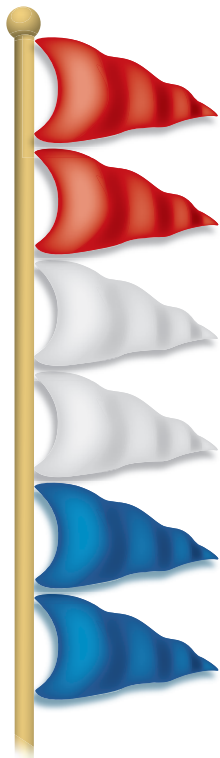
- Dividing $n!$ by $a!, b!, c!$ and so on deals with the effect of repetition caused by objects in the set that are identical. It eliminates arrangements that are the same and that would otherwise be counted multiple times.

CHECK Your Understanding

1. Evaluate the following expressions.

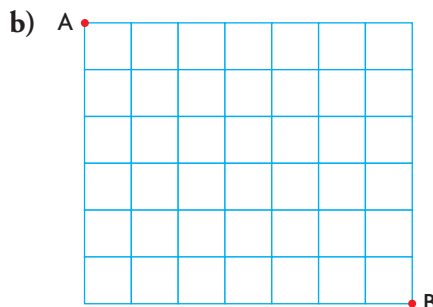
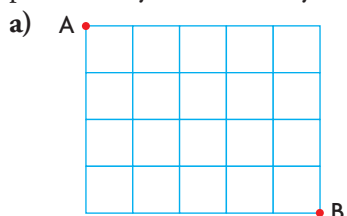
a) $\frac{7!}{3! \cdot 2!}$ b) $\frac{8!}{2! \cdot 2! \cdot 2!}$ c) $\frac{10!}{4! \cdot 3! \cdot 2!}$ d) $\frac{12!}{2! \cdot 4! \cdot 5!}$

2. How many different signals can be made from 6 flags hung in a vertical line, using 2 identical white flags, 2 identical red flags, and 2 identical blue flags?
3. Six nickels are flipped simultaneously. How many ways can three coins land as heads and three coins land as tails?

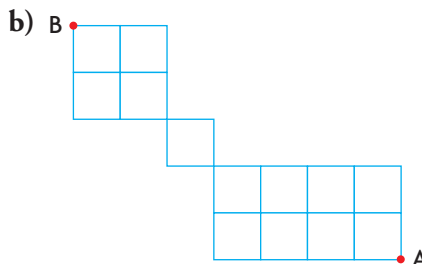
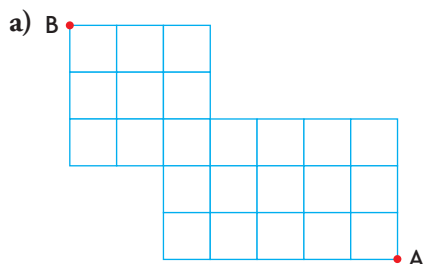


PRACTISING

4. A hockey team has a record of 10 wins, 5 losses, and 3 ties in 18 games. How many different ways could this record have occurred?
5. Norm bought 3 chocolate-chip cookies, 2 peanut-butter cookies, and 4 oatmeal cookies from the corner bakery to give to his 9 grandchildren. How many ways can he distribute 1 cookie to each grandchild?
6. How many different arrangements can be made using all the letters in each word?
 - a) YUKON c) MANITOBA
 - b) ALBERTA d) SASKATCHEWAN
7. A clerk at a bookstore is restocking a shelf of best-selling novels. He has five copies each of three different novels.
 - a) How many different ways can he arrange the books on the shelf?
 - b) How many different ways can these books be arranged on the shelf if the copies of the same novel must be grouped together?
8. Create a counting problem that can be solved using the expression $\frac{8!}{2! \cdot 4!}$.
9. Determine the number of routes there are to get from point A to point B, if you travel only south or east.



10. Jess always walks to her friend's house, which is eight blocks north and five blocks west of her house. How many different routes can she take if she always walks either north or west?
11. How many different routes are there from A to B, if you travel only north or west?



12. A true–false test has eight questions. How many different permutations of answers can the teacher create if five answers are true and three answers are false?
13. The Chief Wakas Totem Pole, located in Vancouver’s Stanley Park, represents the talking stick and characters in an Owikeno (Kwakiutl) story. In the photograph below, the Chief Wakas Totem Pole is on the right. The figures, from the top down, are Thunderbird, Killer Whale, Wolf, Wise One, Huxwhukw (a mythical bird), Bear, and Raven.



The original Chief Wakas Totem Pole (a replica is pictured, far right) was built as a ceremonial entrance to Chief Wakas's house in Alert Bay, Vancouver Island.

- a) Suppose a new totem pole is created using different permutations of all seven figures. How many different arrangements of these figures on the pole are possible?
- b) Suppose another new totem pole is to include these seven figures: two of Thunderbird and two of Wolf, as well as Killer Whale, Wise One, and Bear. How many ways can these figures be arranged on a totem pole?
14. Explain why ${}_nP_n = n!$ cannot be used to determine the number of arrangements of a group of n items when there are a identical items in the group and $a < n$.

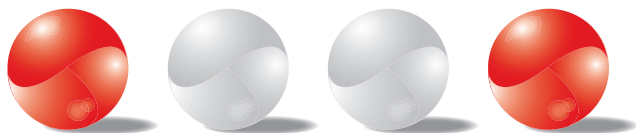
15. a) How many ways can 9 coins (3 dimes, 4 quarters, and 2 loonies) be arranged in a line? State any assumptions you are making.
b) What if the line must begin and end with a loonie? State any assumptions you are making.
16. Sam, Nunzio, and 8 of their friends are playing outside on a hot summer day. How many ways can 10 freezies (3 grape, 2 lime, and 5 orange) be distributed among the 10 children if Sam must have lime and Nunzio must have grape?
17. How many permutations are possible using all the letters of the word STATISTICS for each condition described below?
 - a) You must start with A and end with C.
 - b) The two I's must be together.

Closing

18. Consider the words BANANAS and BANDITS. Explain why there are 12 times the number of permutations possible using all the letters of BANDITS compared to the number of permutations possible using all the letters of BANANAS.

Extending

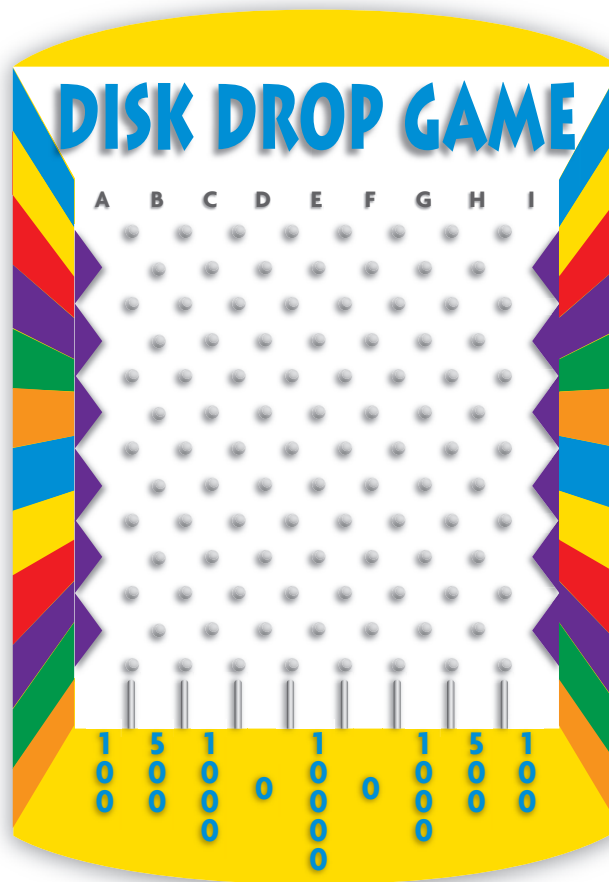
19. A Rubik's cube can be thought of as a grid in three dimensions. How many routes are there from the top rear vertex of the cube to the lower front vertex of the cube, if each route must be as short as possible and follow the grid lines?
20. How many ways can 20 soccer players on a travelling team be assigned to hotel rooms for each situation?
 - a) There are only 10 double rooms.
 - b) There are 5 quadruple rooms.
21. A bag contains three identical red marbles and three identical white marbles. Four marbles are drawn out of the bag and arranged in a row from left to right.
 - a) How many different arrangements might be made?
 - b) What is the likelihood that the arrangement is, from left to right, red, white, white, red?



Applying Problem-Solving Strategies

Analyzing Puzzles and Games: Disk Drop

A television game show offers contestants a chance to win cash by dropping a round disk into the top of a game board from position A through I. The disk falls through the maze to reach one of the nine slots at the bottom. Where it lands determines the value of the cash prize. Counting techniques can be used to analyze this game.



Analyzing the Game

- Down how many different paths can the disk fall to win the grand prize of \$10 000?
- Down how many different paths can the disk fall to win some money, other than the grand prize?
- Do you think this is a fair game for the contestant? Explain.

2.5

Exploring Combinations

GOAL

Explore how counting combinations differs from counting permutations.

YOU WILL NEED

- calculator

EXPLORE the Math

The school athletic council has five members: Jill, Ted, Rhaj, Yvette, and Martin. This Wednesday, they plan to hold a bake sale.



- ? How can you count the number of committees of at least two people that can be chosen to sell baked goods during lunch?

Reflecting

- Share your results and your strategy with several other pairs of students. Discuss any differences.
- When you considered all the possible committees, was each committee a permutation or a **combination** of people? Explain.
- The expression $\frac{5!}{a!b!}$ can be used to determine the number of committees that can be formed from 5 people. What do a and b represent in this expression? Verify that this expression works for all the 2-, 3-, 4- and 5-person committees you considered.
- Suppose only 1 person is needed for the bake sale. How many ways are there to choose that person? Does the expression in part C work in this situation?
- What is the relationship between the number of people to choose from, in this case 5 people, and the variables a and b in the expression $\frac{5!}{a!b!}$?

combination

A grouping of objects where order does not matter. For example, the two objects a and b have one combination because ab is the same as ba .

In Summary

Key Ideas

- When order does not matter in a counting problem, you are determining combinations. For example, abc , acb , bac , bca , cab , and cba are the six different permutations of the letters a , b , and c , but they all represent the same single combination of letters.
- When all n objects are being used in each combination, there is only one possible combination.

Need to Know

- From a set of n different objects, there are always fewer combinations than permutations when selecting r of these objects where $r \leq n$. For example, the number of permutations, P , possible using two of the three letters a , b , and c is

$$P = \frac{3!}{(3-2)!}$$
$$P = 6$$

The six permutations are ab , ba , ac , ca , bc , and cb . However, of these six permutations, only ab , ac , and bc are different two-letter combinations.

FURTHER Your Understanding

1. Brian, Rachele, and Linh volunteer at the food bank on Saturdays.
 - a) In the morning, each person is needed for a different job: stacking cans, stocking dry goods, and cleaning fruits and vegetables. In how many different ways can they be chosen for these jobs?
 - b) List all the ways the three volunteers can be assigned the jobs in part a).
 - c) In the afternoon, all three volunteers are needed to unload vehicles arriving from food drives. In how many ways can they be chosen for this job?
 - d) In the situations above, which involved permutations and which involved combinations? Explain how you know.
2. Explain the main difference between the following:
 - Permutations of 4 objects out of a group of 6 different objects
 - Combinations of 4 objects out of a group of 6 different objects
3. There are 10 members of student council. How many ways can 4 of the members be chosen to serve on the dance committee?
4. There are 12 dogs at the local animal shelter. To increase the likelihood that the dogs will be adopted, 3 of them will appear on a TV morning show. How many ways can 3 of the 12 dogs be selected to appear?



2.6

Combinations

GOAL

Solve problems involving combinations.

LEARN ABOUT the Math

Each year during the Festival du Voyageur, held during February in Winnipeg, Manitoba, high schools compete in the Voyageur Snow Sculpture Contest. This year Amir's school will enter a three-person team. Nine students have volunteered to be on the team.



YOU WILL NEED

- calculator
- standard deck of playing cards

EXPLORE...

- Five cards are dealt to each person in a card game. How many ways can you be dealt a hand that has only red cards?

- ? In how many ways can a team of three snow sculptors be chosen to represent Amir's school from the nine students who have volunteered?

EXAMPLE 1

Calculating combinations

Determine the number of three-person teams that can be formed from the nine volunteers.

Amir's Solution

	First Choice	Second Choice	Third Choice
Team A	Bill	Amir	Connie
Team B	Bill	Connie	Amir
Team C	Amir	Bill	Connie
Team D	Amir	Connie	Bill
Team E	Connie	Bill	Amir
Team F	Connie	Amir	Bill

I started by trying to list all the possible three-person teams. I started by just considering three of the nine possible students. I realized that teams A to F are really the same team. In this situation, the order in which the members of the team are chosen doesn't matter.

$${}_nP_r = \frac{n!}{(n-r)!}$$

where $n = 9$ and $r = 3$

$${}_9P_3 = \frac{9!}{(9-3)!}$$

$${}_9P_3 = \frac{9!}{6!}$$

$${}_9P_3 = \frac{9 \cdot 8 \cdot 7 \cdot 6!}{6!}$$

$${}_9P_3 = 9 \cdot 8 \cdot 7$$

$${}_9P_3 = 504$$

Therefore, 504 teams of three can be formed when the order in which they are chosen matters.

$${}_3P_3 = 3!$$

$${}_3P_3 = 3 \cdot 2 \cdot 1$$

$${}_3P_3 = 6$$

$$\frac{{}_9P_3}{{}_3P_3} = \frac{504}{6}$$

$$\frac{{}_9P_3}{{}_3P_3} = 84$$

Therefore, 84 different teams of three can be formed.

I knew I could calculate the number of permutations involving three people from nine people using the permutation formula.

But 504 is the number of teams when order matters. In this situation, I want to consider only the number of combinations.

The teams listed on the previous page involving Bill, Amir, and Connie are the same because the order in which each team member is selected doesn't matter. These three students can be arranged in 3! or 6 different ways. This is true regardless of which three students are chosen.

I divided the number of permutations, 504, by the number of arrangements that are the same, 6. This gave me the number of combinations.

Reflecting

- Why was it necessary to divide ${}_9P_3$ by $3!$?
- Express $\frac{{}_9P_3}{{}_3P_3}$ in terms of factorials.
- Hanna claims that to determine the number of combinations of r objects chosen from a set of n different objects, you divide ${}_nP_r$ by $r!$. Do you agree? Explain.
- Write a formula you could use to determine ${}_nC_r$, representing the number of combinations possible in which r objects are chosen from a set of n different objects.

Communication | Notation

${}_nC_r$ or $\binom{n}{r}$ are notations commonly used to represent the number of combinations that can be made from a set of n different objects where only r of them are used in each grouping, and $0 \leq r \leq n$.
 ${}_nC_r$ and $\binom{n}{r}$ are read as "n choose r."

APPLY the Math

EXAMPLE 2

Solving a simple combination problem

A restaurant serves 10 flavours of ice cream. Danielle has ordered a large sundae with three scoops of ice cream. How many different ice-cream combinations does Danielle have to choose from, if she wants each scoop to be a different flavour?



Danielle's Solution

Each of these is the same sundae:

<u>V C S</u>	<u>S C V</u>
<u>C V S</u>	<u>S V C</u>
<u>V S C</u>	<u>C S V</u>

I started to visualize all the ways the scoops could be placed in the dish and then realized that the position of each scoop in the dish doesn't matter. For example, one scoop of vanilla, one scoop of chocolate, and one scoop of strawberry is the same sundae no matter how the scoops are arranged.

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

where $n = 10$ and $r = 3$

Since order doesn't matter, I knew I needed to determine the number of combinations using only 3 flavours out of a possible 10.

$${}_{10}C_3 = \frac{10!}{3!(10-3)!}$$

$${}_{10}C_3 = \frac{10!}{3! \cdot 7!}$$

$${}_{10}C_3 = \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{3 \cdot 2 \cdot 1 \cdot \cancel{7!}}$$

I simplified by dividing both the numerator and denominator by 7!.

$${}_{10}C_3 = \frac{10 \cdot \cancel{9} \cdot \cancel{8}}{\cancel{3} \cdot \cancel{2} \cdot 1}$$

I divided 9 by 3 and 8 by 2.

$${}_{10}C_3 = 10 \cdot 3 \cdot 4$$

$${}_{10}C_3 = 120$$

There are 120 different three-scoop sundaes with a different flavour for each scoop.

Your Turn

Danielle's favourite flavour is chocolate. If one scoop in her large sundae must be chocolate and the other two must be different flavours, how many combinations of ice cream are possible?

EXAMPLE 3**Solving a combination problem using the Fundamental Counting Principle**

Tanya is the coach of a Pole Push team that consists of nine players: five male and four female. In each competition, teams of four compete against each other to push their competitors out of a circle. The team that is successful wins.

- How many different four-person teams does Tanya have to choose from for an all-male competition?
- How many different four-person teams does Tanya have to choose from, with two males and two females, for a mixed competition?



Pole Push is played during the Arctic Winter Games. This team from the Northwest Territories placed second in the junior female competition at the 2006 Games, held in Kenai, Alaska.

Tanya's Solution

$$\text{a) } \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

where $n = 5$ and $r = 4$

$$\binom{5}{4} = \frac{5!}{4!(5-4)!}$$

$$\binom{5}{4} = \frac{5!}{4! \cdot 1!}$$

$$\binom{5}{4} = \frac{5 \cdot 4!}{4!}$$

$$\binom{5}{4} = 5$$

I knew that I needed to select four men from the five who were available. Since I was selecting a team of four men, and there were no specific positions within each team, I knew that order doesn't matter. That meant I could use the combination formula.

There are five different all-male teams that can be chosen from the nine players.

b) Men:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

where $n = 5$ and $r = 2$

Women:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

where $n = 4$ and $r = 2$

Let T represent the number of teams:

$$T = \binom{5}{2} \binom{4}{2}$$

$$T = \frac{5!}{2!(5-2)!} \cdot \frac{4!}{2!(4-2)!}$$

$$T = \frac{5!}{2! \cdot 3!} \cdot \frac{4!}{2! \cdot 2!}$$

$$T = \frac{5 \cdot 4 \cdot 3!}{2! \cdot 3!} \cdot \frac{4 \cdot 3 \cdot 2!}{2! \cdot 2!}$$

$$T = 10 \cdot 6$$

$$T = 60$$

There are 60 different four-person teams of two men and two women that can be chosen from the nine players.

I knew that I needed to select two men from the five who were available and two women from the four who were available. Since I was selecting the team and not assigning positions, order did not matter, making this a combination problem.

Since I was selecting men AND selecting women to create the team of four, I multiplied the two combinations using the Fundamental Counting Principle.

Your Turn

How many different four-person teams does Tanya have to choose from for an all-women competition?

EXAMPLE 4**Solving a combination problem by considering cases**

A planning committee is to be formed for a school-wide Earth Day program. There are 13 volunteers: 8 teachers and 5 students. How many ways can the principal choose a 4-person committee that has at least 1 teacher?

Jarrold's Solution: Using direct reasoning

Case 1: 1 teacher and 3 students

$${}_8C_1 \cdot {}_5C_3 = \frac{8!}{1! \cdot 7!} \cdot \frac{5!}{3! \cdot 2!}, \text{ or } 80$$

Case 2: 2 teachers and 2 students

$${}_8C_2 \cdot {}_5C_2 = \frac{8!}{2! \cdot 6!} \cdot \frac{5!}{2! \cdot 3!}, \text{ or } 280$$

Case 3: 3 teachers and 1 student

$${}_8C_3 \cdot {}_5C_1 = \frac{8!}{3! \cdot 5!} \cdot \frac{5!}{1! \cdot 4!}, \text{ or } 280$$

Case 4: 4 teachers and 0 students

$${}_8C_4 \cdot {}_5C_0 = \frac{8!}{4! \cdot 4!} + \frac{5!}{0! \cdot 5!}, \text{ or } 70$$

Number of committees = $80 + 280 + 280 + 70$

Number of committees = 710

The principal can choose from 710 different Earth-Day committees that include at least 1 teacher.

Order does not matter when choosing people for a committee that has no assigned positions, so I knew this problem involves combinations.

I considered the different makeup of the possible 4-person committees, including at least 1 teacher each time, as four different cases.

I evaluated the expression for each case using a calculator.

I used direct reasoning to solve the problem. I dealt directly with each of the cases meeting the "at least 1 teacher" condition.

Since Case 1 OR Case 2 OR Case 3 OR Case 4 are all possible but are exclusive, I added the number of combinations for all four cases.

Shelby's Solution: Using indirect reasoning

Number of committees with no conditions:

$$\binom{13}{4} = \frac{13!}{4! \cdot 9!}$$

Number of committees with 0 teachers and 4 students:

$$\binom{8}{0} \binom{5}{4} = \frac{8!}{0! \cdot 8!} \cdot \frac{5!}{4! \cdot 1!}$$

Number of committees with at least 1 teacher:

$$\binom{13}{4} - \binom{8}{0} \binom{5}{4} = 710$$

The principal can choose from 710 different Earth Day committees that include at least 1 teacher.

I knew that if I figured out how many 4-person committees are possible from a group of 13 with no conditions, then I could just subtract the number of 4-person committees that have 0 teachers (which is the same as the number of committees with 4 students). The difference will be the committees that have at least 1 teacher.

I used indirect reasoning to solve the problem. I did not deal directly with the cases that meet the "at least 1 teacher" condition. I dealt with the other case instead and subtracted.

Your Turn

- Compare Jarrod's and Shelby's solutions. Which solution do you think is more efficient? Explain.
- Would you use direct reasoning or indirect reasoning to solve the following problem: How many ways can the principal choose a four-person committee that has at least one student?

In Summary

Key Ideas

- You can solve counting problems where order is not important by calculating the number of combinations.
- The number of combinations from a set of n different objects, where only r of them are used in each combination, can be denoted by ${}_nC_r$ or $\binom{n}{r}$ and is calculated using the formula

$${}_nC_r = \frac{n!}{r!(n-r)!}, \text{ where } 0 \leq r \leq n$$

For example, if you have a set of three objects, a , b , and c , but you use only two in each combination, the number of combinations is

$${}_3C_2 = \frac{3!}{2!(3-2)!} \text{ or } 3$$

Need to Know

- The formula for ${}_nC_r$ is the formula for ${}_nP_r$ divided by $r!$. Dividing by $r!$ eliminates the counting of the same combination of r objects arranged in different orders.
- When solving problems involving combinations, it may also be necessary to use the Fundamental Counting Principle.
- Sometimes combination problems can be solved using direct reasoning. This occurs when there are conditions involved. To do this, follow the steps below:
 - Consider only cases that reflect the conditions.
 - Determine the number of combinations for each case.
 - Add the results of step 2 to determine the total number of combinations.
- Sometimes combination problems that have conditions can be solved using indirect reasoning. To do this, follow these steps:
 - Determine the number of combinations without any conditions.
 - Consider only cases that do not meet the conditions.
 - Determine the number of combinations for each case identified in step 2.
 - Subtract the results of step 3 from step 1.



CHECK Your Understanding

- Joe has a choice of four flavours of ice cream for his two-scoop sundae: vanilla, strawberry, chocolate, and butterscotch.
 - List all the permutations for a two-flavour sundae.
 - List all the combinations for a two-flavour sundae.
 - How is the number of two-flavour permutations related to the number of two-flavour combinations? Explain.
- From a group of five students, three students need to be chosen for a car-wash committee.
 - How many committees are possible?
 - How many committees are possible, if only two students are needed on the committee?
 - Compare your answers for parts a) and b). What do you notice? Explain why this occurred.
- How many ways can 6 people be selected from a group of 12 to form a dodge-ball team?

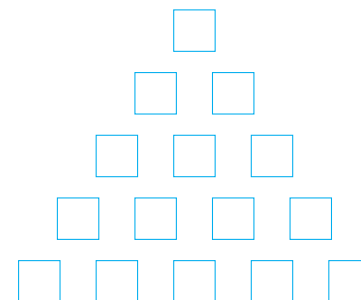
PRACTISING

- Evaluate the following.

a) ${}_5C_3$	c) $\binom{6}{4}$	e) $\binom{12}{6}$
b) ${}_9C_8$	d) ${}_{10}C_0$	f) ${}_8C_1$
- How many ways can 6 players be chosen to start a volleyball game from a team of 10?
- An online music store offers 5 free songs when you join. It has 55 hip-hop songs available. How many different combinations of hip-hop songs can you download for free?
- The card game Crazy Eights is played with a standard deck of playing cards. How many different 8-card hands can be dealt?
- Connie's softball team has 15 players. How many ways can the coach choose his starting lineup of 9 players, if Connie must be the pitcher?
 - Does this problem involve permutations or combinations? Explain.
 - Solve the problem.
- Marnie claims that $\binom{6}{2} = \binom{6}{4}$. Do you agree? Justify your decision.
 - Examine several more cases with the same relationship as part a). What do you notice?
 - Based on your observations in part b), suggest a relationship between $\binom{n}{r}$ and $\binom{n}{n-r}$ for the natural numbers n and whole numbers r .

10. Suppose that 5 teachers and 8 students volunteered to be on a graduation committee. The committee must consist of 2 teachers and 3 students. How many different graduation committees does the principal have to choose from?
11. How many 5-person committees can be formed from a group of 6 women and 4 men, under each of the following conditions.
- There are no conditions.
 - There must be exactly 3 women.
 - There must be exactly 4 men.
 - There can be no men.
 - There must be at least 3 men.
12. A youth hostel has 3 rooms that contain 5, 4, and 3 beds, respectively. How many ways can 12 students be assigned to these rooms?
13. a) For each expression, state the number of different objects in the set and how many are used in each combination.
- i) ${}_nC_r = \frac{5!}{3!(5-3)!}$ ii) ${}_nC_r = \frac{10!}{2!(10-2)!}$ iii) ${}_nC_r = \frac{5!}{3!2!}$
- b) Choose one expression from part a) and create a combination problem that could be solved using that expression.

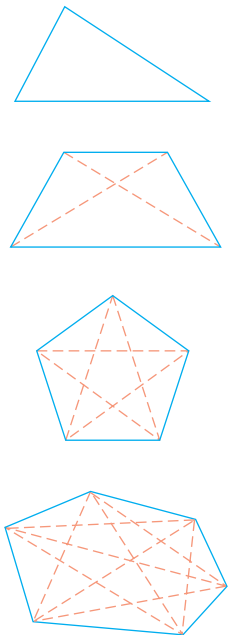
14. Pascal's triangle can be created using the combinations below.
- Evaluate each combination.
 - ${}_0C_0$
 - ${}_1C_0, {}_1C_1$
 - ${}_2C_0, {}_2C_1, {}_2C_2$
 - ${}_3C_0, {}_3C_1, {}_3C_2, {}_3C_3$
 - ${}_4C_0, {}_4C_1, {}_4C_2, {}_4C_3, {}_4C_4$
 - Copy the triangular arrangement of boxes at right. Write each answer from part a) in a box in order, starting with the answer to part i) at the top.
 - In parts a) and b), you created the first five rows of Pascal's triangle. Describe at least two patterns you observe in the triangle.
 - Write two more rows of Pascal's triangle using the patterns you observed.
 - How does Pascal's triangle relate to the pathway problems in Lesson 4.4?



15. Solve each equation. State any restrictions on the variable.

a) ${}_nC_2 = 15$ c) $4\binom{n}{2} = \binom{n+2}{3}$

b) ${}_nC_4 = 35$ d) $\binom{6}{r} = 15$



16. A children's hospital in a city of about one million people is running a charity lottery called Lucky Six to raise money. Players choose six numbers from the numbers 1 to 66. The player wins if the six numbers chosen match six numbers drawn at random by the organizers.
- How many ways could the player win?
 - How many ways could the player lose?
 - Is this a reasonable game for the hospital to run? Explain.
17. How can the combination formula be used to determine the number of diagonals in an n -sided polygon?
18. There are 7 boys and 13 girls in the school art club. A group of 5 is needed to set up an art exhibit. How many different groups of 5 students with at least 2 boys are there to choose from?
- Solve the problem using direct reasoning.
 - Solve the problem using indirect reasoning.
 - Which approach do you prefer? Explain why.

Closing

19. a) How are combinations and permutations similar? How are they different? Use examples in your answers.
- b) If you know the value of ${}_nP_r$, how can you determine the value of ${}_nC_r$? Use examples in your answer.

Extending

20. A CD player holds five different CDs. The CD player is set on shuffle so it randomly selects songs to play from the five CDs. This chart shows the number of songs there are on each CD.

CD Number	1	2	3	4	5
Number of Songs	12	14	15	12	18

Determine the probability that each of the following events will happen during the first five songs played.

- The five songs will be from CD 2 and CD 4.
 - One of the five songs will be from each CD.
 - Your favourite song from each of the 5 CDs will be played.
21. Simplify: $\binom{n}{3} + \binom{n}{2} + \binom{n}{1}$
22. Prove: ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$

2.7

Solving Counting Problems

GOAL

Solve counting problems that involve permutations and combinations.

INVESTIGATE the Math

A band has recorded 3 hit singles over its career. One of the hits went platinum. The band members selected 15 songs, including the 3 hits, for the set list of their upcoming concert.



? How many possible sequences for the set list are there if the band wants to open with a hit, play a hit in the middle of the show, and end the concert with their platinum hit?

- A. Does the order of the songs matter in each possible set list? Explain.
- B. Are there any conditions that have to be met? Explain.
- C. How many platinum hit songs are there to choose from for the song in the 15th position?
- D. How many hit songs are there to choose from for the song in the 1st position?
- E. Once they have chosen the hit song for the 1st position, how many hit songs are there to choose from for the song in the middle (8th) position?
- F. How many ways can they arrange the remaining songs around their hits?
- G. Write an expression that represents the number of different set lists that meet all the conditions. Evaluate your expression.

YOU WILL NEED

- calculator
- standard deck of playing cards

EXPLORE...

- There are five swimmers in the first heat of a race: Aubrey, Betz, Cam, Deanna, and Elena.
 - a) How many ways can the five swimmers finish first, second, and third?
 - b) How many ways can the five swimmers qualify for the final race if the top three finishers qualify?

Reflecting

- H. Did your expression in part G involve combinations or permutations?
- I. Did you use the Fundamental Counting Principle to create your expression in part G? Explain.
- J. Suppose the band had only two hits and wanted to play one at the beginning and one at the end of the concert. Would this increase or decrease the number of possible set lists? By what factor would it change the number of possible set lists?

APPLY the Math

EXAMPLE 1 Solving a permutation problem with conditions

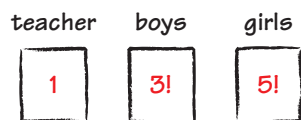
A piano teacher and her students are having a group photograph taken. There are three boys and five girls. The photographer wants the boys to sit together and the girls to sit together for one of the poses. How many ways can the students and teacher sit in a row of nine chairs for this pose?

Yvette's Solution



In this problem, I knew that order matters, since sitting in a different order creates a different arrangement, or permutation.

I drew a diagram with three boxes to help me visualize the problem: one box for the teacher, one box for the boys, and one box for the girls. I did this to represent all the boys sitting together and all the girls sitting together.



I knew that if I decided to place the teacher in the first box, this can happen in only one way.

In the box for boys, I knew there are $3!$ permutations. In the box for girls, I knew there are $5!$ permutations.



Then I realized that the three boxes themselves could be arranged in $3!$ or 6 ways.

Let S represent the number of seating arrangements:

$$S = (1 \cdot 3! \cdot 5!) \cdot 3!$$

$$S = 1 \cdot 6 \cdot 120 \cdot 6$$

$$S = 4320$$

There are 4320 different ways the students and teacher can sit in a single row of nine chairs.

I used the Fundamental Counting Principle to write an expression to represent all the possible seating arrangements, since we are arranging the boys within one box AND arranging the girls within one box AND arranging the position of the three boxes that represent the teacher, the boys, and the girls.

Your Turn

For another pose, the photographer wants the two tallest students, Jill and Sam, to sit at either end, Jill on the left and Sam on the right, and the teacher to sit in the middle. How many different seating arrangements are there for this pose?

EXAMPLE 2

Solving a combination problem involving multiple choices

Combination problems are common in computer science.

Suppose there is a set of 10 different data items represented by $\{a, b, c, d, e, f, g, h, i, j\}$ to be placed into four different memory cells in a computer. Only 3 data items are to be placed in the first cell, 4 data items in the second cell, 2 data items in the third cell, and 1 data item in the last cell. How many ways can the 10 data items be placed in the four memory cells?

Ariston's Solution

3 of 10 items in cell 1: $\binom{10}{3}$

Since the order in which the items are placed within each memory cell does not matter, I knew that the problem involves combinations.

I started by thinking about the first cell and how many ways 3 of 10 items could be placed there.

4 of the remaining 7 items in cell 2: $\binom{7}{4}$

That left 7 items, of which 4 had to go in the second cell.

2 of the remaining 3 items in cell 3: $\binom{3}{2}$

Of the 3 items remaining, 2 had to go in the third cell.

1 item in cell 4: $\binom{1}{1}$

That left 1 item for the last cell.

Let M represent the number of ways to place all 10 items in cells 1, 2, 3, and 4:

$$M = \binom{10}{3} \binom{7}{4} \binom{3}{2} \binom{1}{1}$$

$$M = 12\,600$$

Since all 10 data items must be placed in the four memory cells (3 items in cell 1 AND 4 items in cell 2 AND 2 items in cell 3 AND 1 item in cell 4), I used the Fundamental Counting Principle.

I used a calculator, since there are several expressions involved.

The 10 data items can be placed in the four memory cells 12 600 ways.

Your Turn

Jim claims that he solved the problem by assigning a cell to each data item. He knew that cell 1 was to get 3 items, cell 2 was to get 4 items, cell 3 was to get 2 items, and cell 4 was to get 1 item. He started to list the possibilities in a chart:

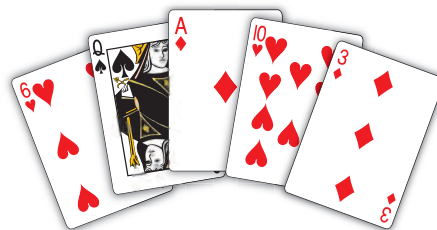
Data Item	a	b	c	d	e	f	g	h	i	j
Cell #	1	1	1	2	2	2	2	3	3	4
Cell #	1	2	3	4	1	2	3	1	2	2
Cell #										

He then realized that each assignment of items to cells corresponded to a different arrangement of the cell numbers. In other words, it was a permutation problem for the numbers 1112222334. Does Jim's line of reasoning result in the same answer?

EXAMPLE 3

Solving a combination problem involving cases

How many different five-card hands that contain at most one black card can be dealt to one person from a standard deck of playing cards?.



Sunny's Solution

Case 1: exactly 1 black and 4 red cards

Case 2: exactly 0 black and 5 red cards

Since the order of the cards in each hand does not matter, I knew that the problem involves combinations.

To satisfy the condition that the hand contains at most 1 black card, I separated the situation into the only two possible cases.

Case 1: exactly 1 black and 4 red
 ${}_{26}C_1 \cdot {}_{26}C_4$

I used the fact that there are 26 black cards and 26 red cards to represent the number of combinations that result in 1 black AND 4 red cards.

Case 2: exactly 0 black and 5 red
 ${}_{26}C_0 \cdot {}_{26}C_5$

I represented the number of combinations that result in 0 black AND 5 red cards.

Let H represent the number of hands with at most 1 black card:

$$\begin{aligned} H &= {}_{26}C_1 \cdot {}_{26}C_4 + {}_{26}C_0 \cdot {}_{26}C_5 \\ H &= 26 \cdot 14\,950 + 1 \cdot 65\,780 \\ H &= 454\,480 \end{aligned}$$

Since the condition “at most 1 black card” is met by case 1 OR case 2, I knew I should add the combinations for each case.

There are 454 480 different five-card hands that contain at most one black card.

Your Turn

Solve the problem above using indirect reasoning. To do this, you will need to consider the total number of five-card hands as well as the number of five-card hands that do not meet the condition “at most one black card.”

In Summary

Key Idea

- When solving counting problems, you need to determine if order plays a role in the situation. Once this is established, you can use the appropriate permutation or combination formula.

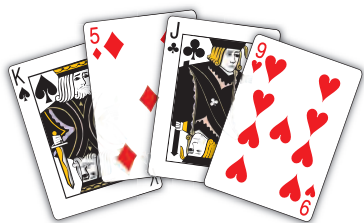
Need to Know

- Once you have established whether a problem involves permutations or combinations you can also use these strategies:
 - Look for conditions. Consider these first as you develop your solution.
 - If there is a repetition of r of the n objects to be eliminated, it is usually done by dividing by $r!$.
 - If a problem involves multiple tasks that are connected by the word AND, then the Fundamental Counting Principle can be applied: multiply the number of ways that each task can occur.
 - If a problem involves multiple tasks that are connected by the word OR, the Fundamental Counting Principle does not apply: add the number of ways that each task can occur. This typically is found in counting problems that involve several cases.

CHECK Your Understanding

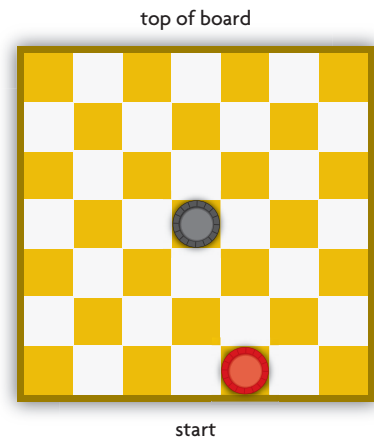
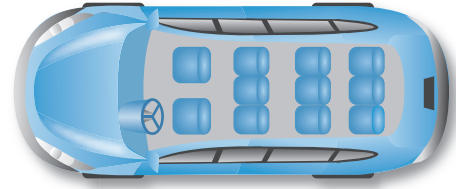
1. Identify whether each situation involves permutations or combinations. Explain how you know.
 - a) Choose 3 toppings for a pizza from 25 different possibilities.
 - b) Choose a CEO, president, and vice-president from a group of 20 candidates.
 - c) Determine the number of outcomes possible when rolling 3 dice: 1 red, 1 blue, and 1 white.
 - d) Determine the number of ways 5 children from a group of 11 can start in a game of pickup basketball.
2. Consider two situations:
 - Situation A: A committee of 3 is to be selected from a group of 10 people.
 - Situation B: An executive committee consisting of a president, a vice-president, and a secretary is to be selected from 10 people.Determine which of these situations involves combinations and which involves permutations. Explain your answer.
3. Maddy arrived at an auction, and there were only 8 items left to bid on. She likes all 8 items but especially likes 3 items. She can afford to have winning bids on only 3 items. How many ways can she bid on 3 items under each of the following conditions?
 - a) She bids on only her 3 favourite items.
 - b) She bids on any 3 of the 8 items.

PRACTISING



4. From a standard deck of 52 cards, how many different four-card hands are there with one card from each suit?
5. How many ways can the top five cash prizes be awarded in a lottery that sold 200 tickets
 - a) if each ticket is not replaced when drawn?
 - b) if each ticket is replaced when drawn?
6. How many ways can the 5 starting positions on a basketball team (1 centre, 2 forwards, and 2 guards) be filled from a team of 2 centres, 4 guards, and 6 forwards?
7. How many ways can five different pairs of identical teddy bears be arranged in two rows of five for a photograph?
8. Five different signal flags fly on the flag pole of a coast guard ship. You can send signals using one or more of the flags. How many different signals can be sent using at least three of these flags?

9. Six different types of boats have pulled into a marina and want to dock at the six available slips. The six slips are adjacent to each other. How many ways can the six boats dock so that the two cabin cruisers, which are travelling together, are docked next to each other?
10. Mark is the coach of a boys' basketball team. He has rented a passenger van as shown to drive to a weekend tournament. How many ways can the 10 players sit in the van if Mark drives and there must be three players on each bench? State any assumptions you are making.
11. a) How many different arrangements are possible for the letters in the word FUNNY if there are no conditions?
b) How many different arrangements are possible if each arrangement must start and end with an N?
12. A combination lock opens when the right sequence of three numbers from 0 to 99 is used. The same number may be used more than once. How many sequences are there that consist entirely of odd numbers?
13. How many different routes can you take to get to a location five blocks south and six blocks east, if you travel only south or east?
14. Three vehicles are taking 16 musicians to a concert: a 5-person car driven by Joe, a 4-person car driven by Kuami, and a 7-passenger van driven by Angela. How many ways can the 16 people be assigned to the 3 vehicles?
15. On the game board shown, the red checker is allowed to move toward the top of the board diagonally left or right. If the black checker is encountered, the red checker cannot move into its square or jump over it. Determine the number of paths the red checker can follow from its starting position to any yellow square along the top of the board.
16. How many different five-card hands that contain at most three hearts can be dealt from a standard deck of playing cards?



Closing

17. Create a flow chart that you could use to help you make decisions about how to solve counting problems.

Extending

18. This week, six boys and seven girls signed up for a ski trip. Only six students can go, so they are to be selected at random. What is the probability that there will be three boys and three girls on the trip?
19. How many four-letter arrangements can be made using the letters in the word ALASKA?
20. How many three-letter arrangements can be made using the letters in the word BOOKS?

History | Connection

Computer Codes

ASCII is an acronym for American Standard Code for Information Interchange. It is a code for representing English characters that appear on computer keyboards.

The ASCII code uses the numbers from 0 to 127. For example, the ASCII code for M is 77. Most computers use ASCII codes to represent text. This makes it possible to transfer text from one computer to another.

All the characters used in email messages are ASCII characters and so are all the characters used in HTML documents. Web browsers read the ASCII characters between carets, < and >, to interpret how to format and display HTML documents.

The standard ASCII codes from 0 to 31 are used to represent control codes for things like Tab, Shift, Esc, and Space Bar. The numbers 32 to 127 are used to represent upper- and lower-case letters, the digits 0 to 9, and other symbols such as #, \$, and ?.

Computers communicate with each other using a binary code system. In this system, there are only two digits (0 and 1), which are represented by two states: off and on. ASCII codes are converted to binary code as a string of 0s and 1s.

- A. Does order matter in the ASCII code system? Explain.
- B. Do you think order matters in the binary code system? Explain.
- C. How long would a string of 0s and 1s in the binary system have to be to represent all of the 128 ASCII codes? Explain how you know.

1. A manufacturer uses a six-character serial number for a line of products. The first and second characters are upper-case letters (A to Z). The third, fourth, and fifth characters are digits (0 to 9). There are only three choices for the last position: A, B, and X.
 - a) How many different serial numbers are possible, if repetition of characters is allowed?
 - b) How many different serial numbers are possible, if no repetition is allowed?
2. How many different ways are there to draw 1 card that is a spade or a diamond from a standard deck of 52 cards?
3. Simplify each expression. State the restrictions on the variable.
 - a) $(n + 10)(n + 9)!$ b) $\frac{(n - 2)!}{n!}$
4. A parking lot attendant has 5 cars to park: 1 blue, 1 white, 1 red, and 2 black.
 - a) How many different ways can the 5 cars be parked side by side?
 - b) How many different ways can the cars be parked so the 2 black cars are next to each other?
5. A book club offers a selection of four books from a list of nine different titles.
 - a) How many different four-book selections can be made?
 - b) How many different four-book selections can be made if the four selections are listed in order of preference?
 - c) Why are the answers to parts a) and b) different? Explain.
6. Solve for n : ${}_nP_4 = 84({}_nC_2)$
7. David and Susan belong to a math club at their school. There are 6 boys and 8 girls in the club. How many different ways can a 5-person committee be selected from the 14 club members under each of the following conditions?
 - a) There must be 2 boys and 3 girls.
 - b) There must be at least 2 boys.
 - c) David and Susan must be on the committee.
 - d) There must be more girls than boys.
8. How many different arrangements are there of the letters in the word TEETH?
9. The nine members, five boys and four girls, of a softball team are arranging themselves in a line for a team photograph. For one of the poses, the photographer wants a boy on either side of each of the four girls. How many different arrangements are possible?

WHAT DO You Think Now? Revisit **What Do You Think?** on page 65. Have your answers and explanations changed?

2

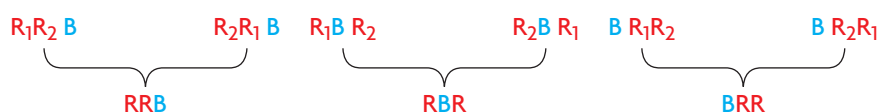
Chapter Review

Study Aid

- See Lesson 2.4, Examples 1 to 3.
- Try Chapter Review Questions 10 and 11.

Q: Why does a set of n objects containing some identical objects have fewer permutations than a set of n different objects?

A: Since all permutations of a set of objects must be different, all repeated arrangements caused by any identical objects in the set have to be eliminated. Think about how you could arrange a set of three tiles: two red and one blue. If you list the possible arrangements using R_1 , R_2 , and B to represent the tiles, you get six arrangements. However, some of those are the same, so you really have only three different arrangements, or permutations.



To determine the number of permutations, determine the number of permutations if the three tiles are all different, $3!$, and divide by $2!$ (2 is the number of objects that are identical). The number of permutations of a set of three tiles with two that are identical can be represented as

$${}_3P_2 = \frac{3!}{2!} \text{ or } 3$$

Study Aid

- See Lesson 2.5 and Lesson 2.6, Examples 1 and 2.
- Try Chapter Review Questions 13 to 16.

Q: Why are there fewer combinations of a set of n objects than permutations of the same n objects?

A: Order does not matter when counting combinations, but it does matter for permutations. Think about how you could arrange a set of three different letters (A , B , and C) in a row of three letters. There are six permutations. However, because order does not matter when you count combinations, those six permutations are all the same combination.

Six permutations:

ABC , ACB , BAC , BCA , CAB , CBA

One combination:

ABC

To determine the number of three-letter combinations for this set, determine the number of permutations, $3!$, and then divide by the number of ways the letters can be arranged, $3!$. The number of three-letter combinations of a set of three letters can be represented as

$${}_3C_3 = \frac{3!}{3!} \text{ or } 1$$

Q: How do you decide what strategy to use to solve a counting problem?

A: First, determine if order is important. If it is, use a permutation model. If not, use a combination model.

- Look for conditions. Consider these first as you develop your solution.
- If there is a repetition of r of the n objects to be eliminated, it is usually done by dividing by $r!$.
- If a problem involves multiple tasks that are connected by the word AND, then the Fundamental Counting Principle can be applied: multiply the number of ways that each task can occur.
- If a problem involves multiple tasks that are connected by the word OR, the Fundamental Counting Principle does not apply: add the number of ways that each task can occur. This typically is found in counting problems that involve several cases.

Study Aid

- See Lesson 2.6, Example 3, and Lesson 2.7, Examples 1 to 3.
- Try Chapter Review Questions 17 to 19.

PRACTISING

Lesson 2.1

1. When is the Fundamental Counting Principle used to solve a counting problem? Use an example in your explanation.
2. Create a tree diagram to show all the possible ways that three coins can land when you flip a quarter, a toonie, and a loonie all at the same time.
3. A student is writing a 10-question multiple-choice test. Each question has 4 choices: A to D. How many different sets of answers can the student give?

Lesson 2.2

4. Solve each equation. State the restrictions on the variable.
 - a) $\frac{(n+2)!}{n!} = 20$
 - b) $\frac{(n+1) \cdot n!}{(n-1)!} = 132$
5. Which expression has a larger value: ${}_6P_6$ or $\frac{8!}{6!}$? Explain how you know.

6. Chorale Saint-Jean from Edmonton, Alberta, is the largest and most active francophone choir in Western Canada. If the singers have rehearsed 12 different songs for an upcoming tour, in how many different orders could they perform the 12 songs?



Lesson 2.3

7. How many different ways can a director of education, a superintendent of curriculum, and a superintendent of finance be selected from group of 25 candidates?
8. A website offers an online practice driving test that consists of 10 questions selected from a bank of 25 questions. The level of each question ranges from easy to difficult. How many different ways can the test be created in each of the following situations?
 - a) There are no conditions.
 - b) The easiest question of the 25 is always first and the most difficult question is always last.
9. Suppose you draw five cards from a standard deck of cards and arrange them in a row from left to right in the order you draw them. How many different five-card arrangements are possible?

Lesson 2.4

10. Consider the 11 letters in the word MATHEMATICS. How many different arrangements are possible in the following situations?
 - a) All the letters are used.
 - b) All the letters are used, but each arrangement must begin with the C.
11. Tina is playing with a tub of building blocks. The tub contains 3 red blocks, 5 blue blocks, 2 yellow blocks, and 4 green blocks. How many different ways can Tina stack the blocks in a single tower in each situation below?
 - a) There are no conditions.
 - b) There must be a yellow block at the bottom of the tower and a yellow block at the top.

Lesson 2.5

12. Which expression results in the greatest value:

$${}_{10}C_5 \text{ or } {}_{11}C_7 \text{ or } \binom{15}{2}?$$

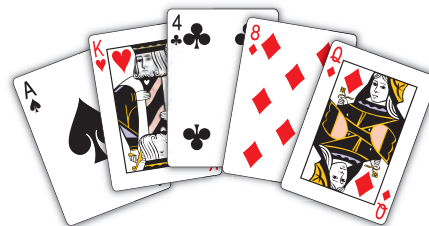
13. How many different selections of 4 books can Ruth choose from a box of 20 different books at her neighbour's garage sale?
14. Liz claims that if you have a set of n different objects and you select r of them, the number of combinations you can make will always be greater than the number of permutations. Do you agree? Justify your decision.

Lesson 2.6

15. The town council is forming a committee, and 9 men and 10 women have volunteered. How many different ways can a committee of 4 people be chosen in each situation below?
 - a) There are no conditions.
 - b) There must be an equal number of men and women on the committee.
 - c) No men can be on the committee.
16. How many different ways can 15 teachers be divided into 3 groups of 5 for an activity at a staff meeting?

Lesson 2.7

17. If 12 points are arranged in a circle, how many different ways can the points be joined to form straight lines?
18. Kim, Kandice, and Kerry are female triplets. They and their 10 cousins are posing for a series of photographs. One pose involves all 13 children. How many ways can the 7 boys and 6 girls be arranged in one row under each of the following conditions?
 - a) The boys and girls must alternate positions.
 - b) The triplets must stand next to each other.
19. How many different five-card hands with at least two face cards can be dealt from a standard deck of playing cards?



Analyzing a Traditional Game

Traditional games that resemble games played with dice are found in many First Nations communities across Canada and are played by men and women. The equivalents of dice vary, but are commonly made of stone, bone, fruit seeds, shell, wood, or reed. They can be in a variety of shapes and are marked in many different ways.

The dice are tossed from the hand or from a small basket, wooden bowl, or cup. How the game is played and scored varies from nation to nation, community to community, even family to family.

One version of this traditional game is played with eight flat bone buttons. One side of each button is plain and the other side is marked with a symbol. Dried beans are used as counters for scoring and are placed in a pile in the centre so that all players can reach them.

- One player tosses all 8 dice.
- The player counts how many land with the same side up.
- If all 8 land with the same side up, the player takes 10 beans.
- If 7 land with the same side up, the player takes 4 beans.
- If 6 land with the same side up, the player takes 2 beans.
- If 5, 4, or 3 land with the same side up, no beans are taken.
- Once all the beans have been taken from the centre pile, players take beans from other players until one player has all the beans.

? How can you use counting techniques to validate the scoring system used in this version of the game?

- A. Does this problem involve permutations or combinations? How do you know?
- B. Determine the number of ways that each outcome of tossing 8 dice, as described above, can occur.
- C. Do you think the way the scores are assigned to the outcomes is fair? Explain.



The Blackfoot and Cree nations use flat sticks with various patterns on them as dice.



Bone buttons with various symbols on them can be used as dice.

Task Checklist

- ✓ Did you show your work?
- ✓ Did you explain your thinking clearly?

Selecting Your Research Topic

To decide what to research, you can start by thinking about subjects and then consider specific topics. Some examples of subjects and topics are shown in the table below.

Subject	Topic
entertainment	• effects of new devices • file sharing
health care	• doctor and/or nurse shortages • funding
post-secondary education	• entry requirements • graduate success
history of the West and North	• relations between First Nations • immigration

It is important to take the time to consider several topics carefully before selecting a topic for your research project. Below is a list of criteria that will help you to determine if a topic you are considering is suitable.

Criteria for Selecting Your Research Topic

- **Does the topic interest you?**

You will be more successful if you choose a topic that interests you.

You will be more motivated to do the research, and you will be more attentive while doing the research. As well, you will care more about the conclusions you draw.

- **Is the topic practical to research?**

If you decide to use first-hand data, can you generate the data in the time available, with the resources you have? If you decide to use second-hand data, are there multiple sources of data? Are these sources reliable, and can you access them in a timely manner?

- **Is there an important issue related to the topic?**

Think about the issues related to the topic. If there are many viewpoints on an issue, you may be able to gather data that supports some viewpoints but not others. The data you collect should enable you to come to a reasoned conclusion.

- **Will your audience appreciate your presentation?**

Your topic should be interesting to others in your class, so they will be attentive during your presentation. Avoid a topic that may offend anyone in your class.

PROJECT EXAMPLE

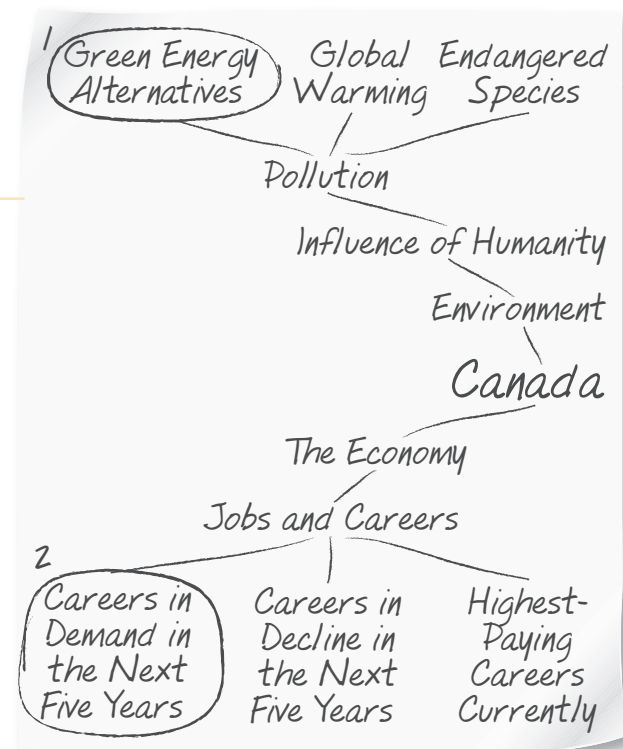
Choosing a topic

Walther identified some current subjects that interest him. Below, he describes how he went from subjects to topics and then to one research topic.

Walther's Research Topic

I identified five subjects that interested me and seemed to be worth exploring: sports, music, post-secondary education, careers, and the environment. I used a mind map to organize my thoughts about each subject into topics. Part of my mind map is shown here.

This gave me a list of possible topics to choose from. I needed to narrow down my list by evaluating each topic. I wanted a current topic that I would enjoy researching and learn something about. I wanted a topic that had accessible data and might interest others in my class. The topic should also involve an issue that would allow me to come to a useful conclusion. I circled the two topics that seemed most likely to work. Then I did an initial search for information about these topics.



1. I spent some time researching Careers in Demand in Canada in the Next 5 Years. I was able to find a lot of websites that predicted which fields would require new employees and the reasons why. However, I realized that finding data to support these claims might be difficult, since these predictions are based on trends. I concluded that this topic would be too difficult to research and support with relevant data.
2. I knew that many organizations around the world such as the World Wildlife Fund track species of animals that are endangered. These organizations keep accurate records and have programs in place to help protect species at risk. As I searched websites for this type of information, I saw that I could find data that supported which species were at risk. I feel that endangered species in Canada is a good topic for my project.

Your Turn

- A. Choose several subjects that interest you. Then make a list of topics that are related to each subject. A graphic organizer, such as a concept web or mind map, is useful for organizing your thoughts.
- B. Once you have chosen several topics, do some research to see which topic would best support a project. Of these, choose the one that you think is the best. Refer to “Criteria for Selecting Your Research Topic.”